Now we can define category $\mathcal{O}$.

<u>Definition</u>. The category $\mathcal{O}_c = \mathcal{O}_c(\Gamma, \mathfrak{h})$ is the category of $H_{1,c}$-modules which are finitely generated over $S\mathfrak{h}^*$ and such that the action of $\mathfrak{h}$ is locally nilpotent. (this is an Abelian category).

<u>Prop. 2.7.</u> Let M be an $H_c = H_{1,c}$-module which is finite over $S\mathfrak{h}^*$. Then $M \in \mathcal{O}_c$ if and only if h acts on M locally finitely.

<u>Pf.</u> First show that if M is any H_c-module and $v \in M$ is a ~~locally~~ nilpotent vector for $\mathfrak{h}$, then h acts on $(S\mathfrak{h})v$ locally finitely. The proof is by induction on $\dim (S\mathfrak{h})v$. $\exists u \in (S\mathfrak{h})v$, $u \neq 0$ such that $\mathfrak{h}u = 0$. Let U be the space of such u. Let $\langle U \rangle < M$ be the submodule gen. by U. Then h acts on U by $-\sum \frac{2c_s}{1-\lambda_s} s + \frac{\dim \mathfrak{h}}{2}$, so locally finitely. Hence, h acts locally finitely on $\langle U \rangle$ (as $\langle U \rangle$ is a quotient of $U \otimes S\mathfrak{h}^*$). On the other hand, if $\bar{v}$ is the image of v in $M/\langle U \rangle$, then $\dim (S\mathfrak{h})\bar{v} < \dim (S\mathfrak{h})v$ (as $U \neq 0$), so by induction assumption h acts locally finitely on it. This implies the "only if".

Prop 2.7a If $M \in \mathcal{O}_c$ then the spectrum of $h|_M$ lies in a finite union of arithmetic progressions $\mu + \mathbb{Z}_+$, and generalized eigenspaces of h are finite dimensional.

Pf. Follows from the proof of Prop 2.7.

Pf of Prop 2.7 cont'd

To prove the "if" part, assume that $\mathfrak{h}$ is locally finite on M, and pick a f.d. $\mathfrak{h}$-invariant generating subspace $E \subset M$ over $\mathfrak{sl}_2^*$, and let μ_j be the eigenvalues of h on E ($j = 1, \ldots, N$). Then the eigenvalues of h on M belong to $\cup_j(\mu_j + \mathbb{Z}_+)$. Thus, $\mathfrak{f}$ acts locally nilpotently on M, since it reduces the eigenvalues of h by 1.

($t=1$)

Prop 2.8. The Verma module $\Delta(\tau)$ has a unique maximal proper submodule $J(\tau)$.

Pf. Let $J(\tau)$ be the sum of all proper submodules of $\Delta(\tau)$. Let h_τ be the eigenvalue of h on the highest wt $\tau \subset \Delta(\tau)$

$h(\tau) = -\sum \frac{2c_s}{1-\lambda_s} s|_\tau + \frac{\dim \mathfrak{h}}{2}$. Then eigenvalues of h on $J(\tau)$ belong to $h(\tau)+1+\mathbb{Z}_+$, so $h(\tau)$ does not occur, and hence $J(\tau)$ is proper. ∎

Define $L(\tau) = \Delta(\tau)/J(\tau)$. Then $L(\tau)$ is simple, and any simple L in $\mathcal{O}_c$ is of the form $L(\tau)$ (since it has "lowest" eigenvalue of h).

Remark. Any f.d. $H_{1,c}$-module belongs to $\mathcal{O}_c$.

Definition. The character of $M \in \mathcal{O}_c$ is $\chi_M(g,q) = \mathrm{Tr}_M(gq^h)$, $g \in \Gamma$.

This makes sense as a series in q (infinite in the positive direction), by Prop 2.7a.

Example. $\chi_{\Delta(\tau)}(g,q) = \dfrac{\chi_\tau(g)\, q^{h(\tau)}}{\det_{\mathfrak{h}^*}(1-gq)}$

Pf. This follows from Molien's formula: if $A: V \to V$ is a linear map then

$$\sum \mathrm{tr}_{S^n V}(S^n A)\, q^n = \frac{1}{\det(1-qA)}.$$

Partial order on weights. Suppose c is fixed. Then we have the function $h(\tau) = h_c(\tau)$, the eigenvalue of h on the highest wt of $\Delta(\tau)$ (or $L(\tau)$):

$$h(\tau) = \frac{\dim \mathfrak{h}}{2} - \sum \frac{2c_s}{1-\lambda_s}\, s.$$

We will say that $\tau \overset{c}{>} \sigma$ if

$h_c(\tau) = h_c(\sigma) - n$, where n is a strictly positive integer.

This defines a partial order on representations of Γ.

Now we want to discuss in more detail the example $\Gamma = G(\ell, 1, n)$. The irreducible representations of $G(\ell, 1, n)$ are parametrized by

multipartitions $\lambda=(\lambda^1, \lambda^2, \dots, \lambda^\ell)$
such that $\sum_i |\lambda^i| = n$
This parametrization is as follows.
Given λ, let $n_i = |\lambda_i|$, and for $1\le j\le \ell$,
let $\chi_j : \mathbb{Z}_\ell \to \mathbb{C}^*$, $\chi_j(k) = \exp\left(\frac{2\pi i jk}{\ell}\right)$.
Now define π_λ to be the representation

$$\pi_\lambda = \mathrm{Ind}_{(S_{n_1}\times\cdots\times S_{n_\ell})\ltimes \mathbb{Z}_\ell^n}^{S_n\ltimes \mathbb{Z}_\ell^n}\left[(\pi_{\lambda_1}\otimes\chi_1^{\otimes n_1})\otimes\cdots\otimes(\pi_{\lambda_\ell}\otimes\chi_\ell^{\otimes n_\ell})\right]$$

<u>Proposition</u> 28. This construction produces all the irreducible representations of $G(\ell,1,n)$, and each exactly once.
<u>Proof.</u> Standard

<u>Example</u>. Consider the case $G(\ell,1,1)=\mathbb{Z}_\ell$.
Then the Cherednik algebra H_c is defined by the relations $sxs^{-1}=\lambda x$, $sys^{-1}=\lambda^{-1}y$,

$$[y,x] = 1 - 2\sum_{j=1}^{\ell-1} c_j s^j. \qquad s^\ell = 1 \qquad \left(\lambda = e^{\frac{2\pi i}{\ell}}\right)$$

Let us consider a more convenient parametrization. Let $e_1, \dots, e_\ell$ be the idempotents of the representations

(characters)

$x_1, \dots, x_\ell$ in $\mathbb{C}\Gamma$. Then we get

$$[y,x] = \sum_{i=1}^{\ell} \mu_i e_i,$$

where $\sum_{i=1}^{\ell} \mu_i = \ell$.

We have standard modules Δ_i with highest weights χ_i. We have $\Delta_i = \mathbb{C}[x]v_i$, and $yx^m v_i = \beta_{m,i} x^{m-1} v_i$, $\beta_{m,i} \in \mathbb{C}$. Let us write recursion for $\beta_{m,i}$.

We have

$$\beta_{m,i} x^{m-1} v_i = yx^m v_i = xyx^{m-1} v_i + \left(\sum \mu_j e_j\right) x^{m-1} v_i$$

$$= \beta_{m-1,i} x^{m-1} v_i + \mu_{i+m-1} x^{m-1} v_i$$

So we get $\beta_{m,i} = \beta_{m-1,i} + \mu_{i+m-1}$, and thus $\beta_{m,i} = \sum_{j=0}^{m-1} \mu_{i+j}$.

Thus we see:

<u>Prop 2.9</u>. Δ_i is irreducible if and only if $\forall m \geq 1$, $\sum_{j=0}^{m-1} \mu_{i+j} \neq 0$.

<u>Example</u>. $\ell = 2$, $[y,x] = 1 - 2cs = \mu_1 e_1 + \mu_2 e_2$, $\mu_1 + \mu_2 = 2$, $\mu_1 = 1 + 2c$, $\mu_2 = 1 - 2c$.

$\Delta_2 = \Delta(\mathbb{C})$ is irreducible iff $\beta_{m,2} \neq 0 \ \forall m$

$$\beta_{m,2} = \begin{cases} m, & m \text{ even} \\ 1-2c+2\left[\frac{m}{2}\right] = m-2c, & m \text{ is odd} \end{cases}$$

So $\Delta(\mathbb{C})$ is irreducible unless

$c = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots,$

$\Delta_1 = \Delta(\mathbb{C}_-)$ is irreducible iff $\beta_{m,1} \neq 0 \quad \forall m$

$$\beta_{m,1} = \begin{cases} m, & m \text{ even} \\ 1+2c+2\left[\frac{m}{2}\right] = m+2c & \text{iff } m \text{ is odd} \end{cases}$$

So $\Delta(\mathbb{C}_-)$ is irreducible unless $c = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$

The simple module $L_i(\mu)$ is $\Delta_i(\mu)$ if $\Delta_i(\mu)$ is irreducible ($\Leftrightarrow \sum_{j=0}^{m-1} \mu_{i+j}$ never vanishes), otherwise L_i is finite dim.

$L_i \cong \mathbb{C}[x]/x^m$, where m is the smallest integer such that $\sum_{j=0}^{m-1} \mu_{i+j} = 0$.

In particular, in the most degenerate case, all L_i except one are finite dimensional.

Prop 2.10 Any $M \in \mathcal{O}_c$ has a finite filtration whose successive quotients are highest weight modules (= quotients of Verma modules).

Pf. Let λ be an eigenvalue of h on M such that $\lambda-1$ is not an eigenvalue. Let $M(\lambda)$ be the eigenspace, and $\tau \subset M(\lambda)$ an irreducible T-submodule. Then we have a homom. $\psi: \Delta(\tau) \to M$ whose image is a highest wt module. Let $F_0 M = \mathrm{Im}\,\psi$, and then consider $M^{(1)} = M/F_0 M$, and apply the same procedure. We get a filtration $F_0 M \subset F_1 M \subset \cdots$ which has highest wt successive quotients $F_{i+1}M/F_i M$. We claim that this filtration is finite (and exhausting). Indeed, there are only finitely many λ which can arise, namely $\lambda = h(\sigma)$ for some σ. Let N be the length of the sum of generalized eigenspaces of M with these eigenvalues as a T-module. Then the length of the filtration is $\leq N$. $\blacksquare$

Cor 2.11 Any $M \in \mathcal{O}_c$ has finite length.

Pf. By Prop. 2.10, it suffices to show that $\Delta(\tau)$ has finite length for any τ. We prove it by induction with respect to τ (under the partial order we defined). We have an exact sequence

$$0 \to J(\tau) \to \Delta(\tau) \to L(\tau) \to 0,$$

so it suffices to prove that $J(\tau)$ has finite length. But By Prop 2.10, $J(\tau)$ has a finite filtration whose successive quotients are highest weight modules with highest weight $\sigma < \tau$. So we are done by the induction assumption (For the base we use minimal elts τ of Irrep Γ, for which $\Delta(\tau)$ are simple).

Contragredient modules. We have a natural contravariant functor $\dagger : \mathcal{O}_c(\Gamma, \mathfrak{h}) \to \mathcal{O}_{\bar{c}}(\Gamma, \mathfrak{h}^*)$, where $\bar{c}(s) = c(s^{-1})$. This functor is defined by the formula $M^\dagger = M^{*\,\omega}_{res}$, where M^*_{res} is the restricted dual (direct sum of duals of generalized $\mathfrak{h}$-eigenspaces), and $\omega: H_c(\Gamma,\mathfrak{h}) \to H_{\bar{c}}(\Gamma,\mathfrak{h}^*)$ is the isomorphism given by the formula $\omega|_{\mathfrak{h}} = \omega|_{\mathfrak{h}^*} = \mathrm{id}$, $\omega(g) = g^{-1}$, $g \in \Gamma$. The objects $\Delta_{c^{-1},\mathfrak{h}^*}(\overset{*}{\tau})^+$ are called costandard and denoted by $\nabla(\tau)$. It's easy to see that they have the same characters as $\Delta(\tau)$, and moreover same composition factors, but the composition factors are arranged in the opposite order, e.g. $\Delta(\tau) \twoheadrightarrow L(\tau)$ but $L(\tau) \hookrightarrow \nabla(\tau)$.

Projective covers. We would like to show that $\mathcal{O}_c$ is a finite category, i.e. every object has a projective cover (= has enough projectives). To this end, consider the modules $\Delta(\tau,n) = H_c \otimes_{S\mathfrak{h}\#\Gamma} (\tau \otimes S\mathfrak{h}/\mathfrak{m}^{n+1})$, where $\mathfrak{m} \subset S\mathfrak{h}$ is the max. ideal. (so $\Delta(\tau,0) = \Delta(\tau)$).

<u>Theorem</u> 2.12. For large enough n, $\Delta(\mathbb{C}\Gamma, n) = \bigoplus_{\tau} \dim\tau \cdot \Delta(\tau, n)$ contains a direct summand which is a projective generator for $\mathcal{O}_c$.

<u>Proof.</u> We have degree generator $\partial : \Delta(\mathbb{C}\Gamma, n) \circlearrowleft$, $([\partial, x_i] = x_i, [\partial, y_i] = -y_i, [\partial, \Gamma] = 0)$, and also $h : \Delta(\mathbb{C}\Gamma, n) \circlearrowleft$. So $h - \partial$ is an endomorphism.

Let $\Sigma = \{ h(\tau), \tau \in \operatorname{Irrep} \Gamma \}$, and let $\Delta^{\Sigma}(\mathbb{C}\Gamma, n)$ be the direct sum of generalized eigenspaces of $h - \partial$ on $\Delta(\mathbb{C}\Gamma, n)$ with eigenvalues in Σ. Then $\Delta^{\Sigma}(\mathbb{C}\Gamma, n)$ is a direct summand in $\Delta(\mathbb{C}\Gamma, n)$. Also,

$$\operatorname{Hom}(\Delta^{\Sigma}(\mathbb{C}\Gamma, n), X) = \{ v \in \bigoplus_{\lambda \in \Sigma} X_{gen}(\lambda) : \mathfrak{m}^{n+1} v = 0 \}$$

For large enough n, the condition

generalized eigenspace

$m^{n+1}v=0$ is vacuous, so
$\mathrm{Hom}(\Delta^{\Sigma}(\mathbb{C}\Gamma, n), X) = \bigoplus_{\lambda\in\Sigma} X_{gen}(\lambda)$.

and thus $\Delta^{\Sigma}(\mathbb{C}\Gamma, n)$ is projective.
Also, it's a generator, since if
$\mathrm{Hom}(\Delta^{\Sigma}(\mathbb{C}\Gamma, n), X)=0$ then
$\bigoplus_{\lambda\in\Sigma} X_{gen}(\lambda)=0$, hence $X=0$ (as X must have a filtration with successive quotients highest weight). ∎

Thus, every simple module $L(\tau)$ has a projective cover $P(\tau)$.

<u>Proposition 2.13</u>. $P(\tau)$ is filtered by $\Delta(\sigma)$. Moreover, $P(\tau) \twoheadrightarrow \Delta(\tau)$, and the kernel of this map is filtered by $\Delta(\sigma)$ with $\sigma>\tau$.

<u>Proof</u>. It is easy to show that
$\mathrm{Hom}(\Delta(\lambda),\Delta(\mu))=0$ for $\lambda\not\leq\mu$, $\mathrm{Hom}(\Delta(\lambda),\Delta(\lambda))=\mathbb{C}$, and $\mathrm{Ext}^1(\Delta(\lambda),\Delta(\mu))=0$ for $\lambda\not<\mu$. This means that if we totally order $\mathrm{Irrep}\,\Gamma$ refining the partial order, $\lambda_1>\lambda_2>\cdots>\lambda_N$,

then any Δ-filtered object X in $\mathcal{O}$ has a canonical filtration with successive quotients

$\Delta(\lambda_1)\otimes E_1, \ldots, \Delta(\lambda_N)\otimes E_N$ (bottom to top)

where E_i are f.d. vector spaces.

This filtration must be preserved by every endomorphism of X, since

$\mathrm{Hom}(\Delta(\lambda_i), \Delta(\lambda_j)) = 0$ for $i<j$.

Thus, if $X\oplus Y$ is standardly filtered, then the projector p_X to X on $X\oplus Y$ preserves this filtration, and so must act by a projector p_i on each E_i.

This implies that X has a standard filtration. It is clear that $\Delta^{\Sigma}(\mathbb{C}\Gamma, n)$ has a standard filtration.

Thus, $P(\tau)$ has a standard filtration.

Also, since $P(\tau)$ is projective, the morphism $P(\tau) \twoheadrightarrow L(\tau)$ lifts to $P(\tau)\to\Delta(\tau)$, which is clearly surjective. Also if $P(\tau) \twoheadrightarrow \Delta(\sigma)$ then $P(\tau)\twoheadrightarrow L(\sigma)$, so $\sigma=\tau$. Thus $\mathrm{Ker}\ P(\tau) \twoheadrightarrow \Delta(\tau)$ has a standard filtration. By the ordering argument above, $\forall \sigma$ s.t. $\Delta(\sigma)$ occurs in the kernel, $\sigma>\tau$ ∎

Highest weight structure of $\mathcal{O}_c$

Recall that a highest weight category ((C)) is an abelian category $\mathcal{C}$, equivalent to category of reps of a f.d. algebra, such that:

1) The simples $L(\lambda)$ are labeled by elements of a poset Λ.

2) For each λ, one is given an object $\Delta(\lambda)$ called the standard object labeled by λ, such that $\mathrm{Hom}(\Delta(\lambda), \Delta(\mu)) \neq 0 \Rightarrow \lambda \leq \mu$ and $\mathrm{End}(\Delta(\lambda)) = \mathbb{C}$.

3) One has: $P(\lambda) \twoheadrightarrow \Delta(\lambda)$ (where $P(\lambda)$ is the projective cover of $L(\lambda)$), and the kernel of this map admits a filtration whose successive quotients are $\Delta(\mu)$ with $\mu > \lambda$.

<u>Theorem</u> 2.14 The standard modules $\Delta_c(\lambda)$ define the structure of a highest weight category on $\mathcal{O}_c$, for the partial ordering defined above.

<u>Proof.</u> Follows from the above ~~Statements 1) and 2) are clear ($\Lambda = \mathrm{Irrep}\,\Gamma$). Also the map~~

~~$\mathrm{Hom}(P(\lambda), \Delta(\lambda)) \to \mathrm{Hom}(P(\lambda), L(\lambda)) \cong \mathbb{C}$~~

~~is an isomorphism, since~~

dim Hom$(P(\lambda), \Delta(\lambda)) = [\Delta(\lambda) : L(\lambda)] = 1$.
So $\exists\, \varphi : P(\lambda) \to \Delta(\lambda)$ which gives rise to $\overline{\varphi} : P(\lambda) \twoheadrightarrow L(\lambda)$. The image of φ therefore can't lie in $J(\lambda)$, so φ is surjective.

The fact that Ker φ is standardly filtered is shown by Ginzburg-Guay-Opdam-Rouquier.

__Thm. 2.15__ The objects $\nabla(\lambda) = \Delta(\lambda)^+$ are the costandard objects for $\mathcal{O}_c$.

__Remark.__ Recall that in a highest weight category $\mathcal{C}$, the costandard objects $\nabla(\lambda)$ are the injective hulls of $L(\lambda)$ in $\mathcal{C}_{\leq \lambda}$, the Serre subcategory of $\mathcal{C}$ spanned by L_μ with $\mu \leq \lambda$.

__Proof.__ By the theory of highest weight categories, $\Delta(\lambda)$ are projective covers of $L(\lambda)$ in $\mathcal{C}_{\leq\lambda}$. So $\mathrm{Ext}^i(\Delta(\lambda), L(\mu)) = 0$, $i \geq 1$. $\forall \mu \leq \lambda$. Dualizing, we get $\mathrm{Ext}^i(L(\mu), \Delta(\lambda)^+) = 0$ $i \geq 1$ $\forall \mu \leq \lambda$. So $\Delta(\lambda)^+$ is injective, hence $\Delta(\lambda)^+ = \nabla(\lambda)$.

The case $\Gamma = S_n$

If $\Gamma = S_n$, then $\mathrm{Irrep}\,\Gamma$ can be naturally identified with the set of partitions. Also, there is only one conjugacy class of reflections (namely, transpositions) so c is just one number. The commutation relations look as follows (for $\mathfrak{h} = \mathbb{C}^n$):

$$H_c(S_n, \mathbb{C}^n) = \mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n \rangle \# S_n$$

modulo:

$$[x_i, x_j] = 0, \qquad [y_i, y_j] = 0,$$

$$[y_i, x_j] = c s_{ij}, \qquad [y_i, x_i] = 1 - c\sum_{j \neq i} s_{ij}.$$

Note that $H_c(S_n, \mathbb{C}^n) = H_c(S_n, \mathbb{C}^{n-1}) \otimes \underset{\substack{\uparrow \\ \text{Weyl algebra}}}{A_1}$

$H_c(S_n, \mathbb{C}^{n-1})$ is the subalgebra generated by S_n, $\overline{x}_i$, $\overline{y}_i$ where $\overline{x}_i = x_i - \frac{1}{n}\sum x_j$, $\overline{y}_i = y_i - \frac{1}{n}\sum y_j$.

<u>Theorem 2.16</u> Category $\mathcal{O}_c$ is semisimple with $P(\lambda) = \Delta(\lambda) = L(\lambda)$ unless $c = \frac{r}{m}$,

$2 \leq m \leq n$, $r \in \mathbb{Z}$, and $(r, m) = 1$.

<u>Proof.</u> Will be given later when we discuss KZ functor. ∎

Let us see how $\mathcal{O}_c$ looks like for "interesting" values of c.

It's clear that $\mathcal{O}_c \cong \mathcal{O}_{-c}$, by using $s_{ij} \to -s_{ij}$, so restrict to the case $c \in \mathbb{Q}$, $c > 0$.

<u>Theorem</u> 2.17. (Rouquier) The structure of $\mathcal{O}_c$ does not depend on r, and only depends on m.

<u>Proof</u>. Will be discussed later.

So it's enough for us to consider $c = \frac{1}{m}$.

<u>Examples for interesting values of c</u>

<u>$n=2$, $c=\frac{1}{2}$</u>. We've discussed this. We get category $\mathcal{C}_1$ from appendix. Simples L_0, L_1, $\dim L_0 = 1$, L_1 inf. dimensional (for $\mathbb{C}^{n-1}$).

<u>$n=3$, $c=\frac{1}{3}$</u> $\quad c\Sigma s$: $\ydiagram{3} \mapsto 1 \quad \ydiagram{2,1} \mapsto 0 \quad \ydiagram{1,1,1} \mapsto -1$.

Get a copy of category $\mathcal{C}_2$.

<u>$n=3$, $c=\frac{1}{2}$</u> $\quad \ydiagram{3} \to \frac{3}{2} \quad \ydiagram{2,1} \mapsto 0 \quad \ydiagram{1,1,1} \mapsto -\frac{3}{2}$.

So have two blocks: One trivial block $\ydiagram{2,1}$ $(\mathcal{C}_0)$ and one nontrivial $(\mathcal{C}_1)$, so $\mathcal{C}_0 \oplus \mathcal{C}_1$.

<u>$n=4$, $c=\frac{1}{4}$</u> $\quad \ydiagram{4} \to \frac{3}{2} \quad \ydiagram{3,1} \to \frac{1}{2} \quad \ydiagram{2,1,1} \to -\frac{1}{2} \quad \ydiagram{1,1,1,1} \to -\frac{3}{2}$

$\ydiagram{2,2} \mapsto 0$

So get $\mathcal{C}_0 \oplus \mathcal{C}_3$.

$n=4,\ c=\frac{1}{3}$ $\quad (4) \mapsto 2 \quad (3,1) \mapsto \frac{2}{3} \quad (2,1,1) \to -\frac{2}{3} \quad (1^4) \to -2$

$(2,2) \to 0$

So get ~~three~~ blocks: $\mathcal{C}_2 \oplus \mathcal{C}_0 \oplus \mathcal{C}_0$.

$n=4,\ c=\frac{1}{2}$ $\quad (4) \mapsto 3 \quad (3,1) \mapsto 1 \quad (2,1,1) \to -1 \quad (1^4) \mapsto -3$

$(2,2) \to 0$.

This is actually the first example when we get something new. In this case all irreducibles are in a single block. Let us calculate the Vermas in terms of the irreducibles ~~in terms of Vermas~~, ~~and vice versa~~:

$$L_{(1^4)} = \Delta_{(1^4)}, \quad L_{(2,1^2)} = \Delta_{(2,1^2)} - \Delta_{(1^4)}$$

$$L_{(2,2)} = \overset{2}{\Delta}_{(2,2)} - \overset{3}{\Delta}_{(2,1^2)} + \overset{1}{\Delta}_{(1^4)}.$$ ← supported in codim 2.

$$L_{(3,1)} = \overset{3}{\Delta}_{(3,1)} - \cancel{\Delta_{(2,1^2)} + \Delta_{(1^4)}} - \Delta_{(2,2)}$$

$$L_{(4)} = \Delta_{(4)} - \Delta_{(3,1)} + \Delta_{(2,2)} + \Delta_{(1,3)} - \Delta_{(1^4)}.$$

$$\Delta_{(1^4)} = L_{(1^4)}, \quad \Delta_{(2,1^2)} = L_{(2,1^2)} + L_{(1^4)}$$

$$\Delta_{(2,2)} = L_{(2,2)} + L_{(2,1^2)}$$

$$\Delta_{(3,1)} = L_{(3,1)}$$

$$\Delta_{1111} = L_{1111}$$
$$\Delta_{211} = L_{211} + L_{1111}$$
$$\Delta_{22} = L_{22} + L_{211}$$
$$\Delta_{31} = L_{31} + L_{22} + L_{211} + L_{1111}$$
$$\Delta_4 = L_4 + L_{31} + L_{211}$$

So we see standards of length 3 and 4.

<u>Theorem.</u> (Berest, E, Ginzburg). For $c>0$, finite dimensional representations of $H_c(S_n, \mathbb{C}^{n-1})$ exist only if $c = \frac{r}{n}$, $(r,n)=1$. If c is such, there exists a unique f.d. irreducible representation, which has no self-extensions. This repr. (is $L_c(\mathbb{C})$) has dimension r^{n-1}, and character
$$\chi(g,q) = q^{\frac{(1-r)(n-1)}{2}} \frac{\det_{\mathfrak{h}}(1-q^r g)}{\det_{\mathfrak{h}}(1-qg)}.$$

<u>Proof.</u> We will only give a construction of this representation. Let
$$f(x_1, \dots, x_n) = \frac{1}{2\pi i} \oint \left((z-x_1)\cdots(z-x_n)\right)^{r/n} dz$$
(integral over a large circle. It's easy to check that the ideal of derivatives $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$ is

invariant under Dunkl operators. Thus $L_c(\mathbb{C}) = \mathbb{C}[x_1,\dots,x_n]\big/\left(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}\right)$ $(\sum x_i = 0)$,

the Jacobi ring of f, is a repr. of $H_c(S_n, \mathbb{C}^{n-1})$. It is easy to check that $\frac{\partial f}{\partial x_1} = \dots = \frac{\partial f}{\partial x_n} = 0 \Rightarrow x = 0$, so we have a complete intersection, and the character formula follows.

Example of a highest weight category:

Appendix to Lecture 3 | 1

Category $\mathcal{C}_N$; $N \geqslant 0$

1) Parabolic category $\mathcal{O}$ for $\mathfrak{gl}_{N+1}$ of type $(N, 1)$ (i.e. integrable for $\boxed{\mathfrak{gl}_N}$) with zero central char

2) $\mathcal{D}_{\mathcal{N}}(\mathbb{P}^N)$ - equiv. $\mathcal{D}$-modules on $\mathbb{P}^N$ under Nilp. group $\mathcal{N}$.

3) The nontrivial block of $\operatorname{Rep} H_c(S_{N+1})$ for $c = \frac{1}{n+1}$.

Simples: $L_0, L_1, \ldots, L_N$, ordering being the natural one backwards

Standards:

$$0 \to L_1 \to \Delta_0 \to L_0 \to 0$$
$$0 \to L_2 \to \Delta_1 \to L_1 \to 0$$
$$\ldots$$
$$0 \to L_N \to \Delta_{N-1} \to L_{N-1} \to 0$$

and $\Delta_N = L_N$.

so $\Delta_i = L_i + L_{i+1}$, $i < N$

$\Delta_N = L_N$

in the Grothendieck group.

Costandards $\nabla_N = \Delta_N^+$

projectives:

$P_0 = \Delta_0$

$$0 \to \Delta_0 \to P_1 \to \Delta_1 \to 0$$

$\ldots$

$$\boxed{\begin{matrix} L_0 \\ L_1 \end{matrix}}$$

$$\boxed{\begin{matrix} & L_1 & \\ L_0 & & L_2 \\ & L_1 & \end{matrix}}$$

$$0 \to \Delta_1 \to P_2 \to \Delta_2 \to 0 \qquad \boxed{\begin{array}{ccc} & L_2 & \\ L_1 & & L_3 \\ & L_2 & \end{array}}$$

$$\cdots\cdots$$

$$0 \to \Delta_{N-1} \to P_N \to \Delta_N \to 0 \qquad \boxed{\begin{array}{c} L_N \\ L_{N-1} \\ L_N \end{array}}$$

Injectives are the same except

$I_0 = \nabla_0 = \Delta_0^+$

<u>Tiltings</u>: $T_N = L_N$, $T_{N-1} = P_N = I_N, \ldots,$

$T_0 = P_1 = I_1$.

The functor $X \to Hom(X, T)$:

$$Hom(\Delta_i, T_j) = \begin{cases} Hom(\Delta_i, I_{j+1}) \cong \mathbb{C}^{[\Delta_i, L_{j+1}]} & \text{if } j < N \\ Hom(\Delta_i, L_N), & j = N \end{cases}$$

$$\dim Hom(\Delta_i, T) = \begin{cases} 1 & i = 0 \\ 2 & 0 < i < N \\ 2 & i = N \end{cases}$$

So $\tilde{\Delta}_0$ has length 1 and $\tilde{\Delta}_i$ has length 2, $i > 0$. $\mathcal{C}^\vee$ is the same category, with opposed ordering on Λ.