Lecture 5

Let $H_c = H_{1,c}(\Gamma_n)$ be the symplectic reflection algebra for the wreath product group $\Gamma_n = S_n \ltimes \Gamma_1^n$, $\Gamma_1 = \Gamma \subset SL_2(\mathbb{C})$. A natural question is to describe the category of finite dimensional representations of H_c. This question is solved for $n=1$, but it is difficult for $n>1$, and so we may ask a simpler question to describe f.d. irreducible representations of H_c and find their dimensions (or Γ_n-structure) This is still too hard, so the subject of this lecture is an even simpler question — how many f.d. irreducible representations does H_c have for a given c? We'll see that already this deceptively simple question has a very nontrivial answer (conjectured in my papers and proved by Bezrukavnikov and Losev), which is expressed in terms of affine Lie algebra representations and their restriction to subalgebras.

We start with noting that for generic c, H_c does not have any f.d. representations

Exercise 5.1. Show that if H_c has a f.d representation then c must belong to a countable union of hyperplanes in the space of parameters.

Hint. Fix the Γ_n-type of the repr. and compute the trace of the main commutation relation. This will give the equation of a hyperplane.

For further discussions, it'll be convenient to change the parametrization of our algebras. Namely, recall that if $n \geq 2$, the parameters for H_c are $c = (k, \{c_\gamma, \gamma \in \Gamma\})$, where $k \in \mathbb{C}$ and $c_\gamma \in \mathbb{C}$ for $\gamma \in \Gamma$, $c_1 = 1$ (a conjugation invariant function). Recall also that to Γ one can attach, using McKay's correspondence, a simply laced simple Lie algebra $\mathfrak{g}$. Let $\mathfrak{h} \in \mathfrak{g}$ be a Cartan, $\hat{\mathfrak{g}}$ the corresponding affine Lie algebra $\hat{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}K$, and $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K$ the Cartan in $\hat{\mathfrak{g}}$. Also recall that the vertices of the affine Dynkin diagram I of $\hat{\mathfrak{g}}$ are labeled by irreps of Γ with the extending vertex 0 labeled by the trivial representation.

For each $i \in I$, set

$$\lambda_i = \frac{1}{|\Gamma|} \mathrm{Tr}\Big(\sum_{\gamma\in\Gamma} c_\gamma \gamma\Big|_{E_i}\Big),$$

where E_i is the irrep of Γ corresponding to $i \in I$. Then set $\lambda = \sum \lambda_i \omega_i$, where $\omega_i \in \hat{\mathfrak{h}}^*$ for $i \in I$ is the fundamental weight of $\hat{\mathfrak{g}}$ corresponding to i. Note that since $c_1 = 1$, we have $(\lambda, \delta) = 1$, where $\delta \in \tilde{\mathfrak{h}}^*$ is the basic imaginary root of $\hat{\mathfrak{g}}$ (here $\tilde{\mathfrak{h}}^*$ is the dual of the extended Cartan $\tilde{\mathfrak{h}} = \hat{\mathfrak{h}} \oplus \mathbb{C}d$, where d is the degree derivation of $\hat{\mathfrak{g}}$). Indeed, $\delta = \sum \dim(E_i)\alpha_i$, so $(\lambda, \delta) = \sum \lambda_i \dim E_i = \frac{1}{|\Gamma|} \mathrm{Tr}\Big(\sum_{\gamma\in\Gamma} c_\gamma \Gamma\Big|_{\mathbb{C}\Gamma}\Big) = c_1 = 1$.

Thus, we can parametrize H by k, λ, where $\lambda \in \hat{\mathfrak{h}}^*$ is such that $(\lambda, \delta) = 1$. We'll write $H = H_{k,\lambda}$.

<u>Example</u> 5.2 If $n=1$ then we don't have k so $H = H_\lambda$. The main commutation relation is $[x, y] = \sum_{i\in I} \frac{\lambda_i}{\dim E_i} e_i$, where e_i are the central idempotents of E_i. Now suppose that H_λ has a f.d. representation E with dimension vector $\alpha = (d_i) = \sum_{i \in I} d_i \alpha_i$.

Then, computing the trace of the main commutation relation in E, we get

$$0 = \sum_{i \in I} \lambda_i d_i = (\lambda, \alpha).$$

In fact one can show that the converse is also true: if $(\lambda, \alpha) = 0$ then $\exists$ a f.d. representation E of H_λ with dimension vector α (exercise: prove this) at least for cyclic Γ

In fact, for each λ, one may consider the set of all α such that $(\lambda, \alpha) = 0$, and they generate a finite singly laced root system Σ_λ which is conjugate to one contained in I, and the simple H_λ-modules correspond to simple roots of this system, so the number of simple modules is the rank of Σ_λ.

Now let's consider the case $n > 1$. I will explain the conjecture I made, proved by Bezrukavnikov and Losev. Let $\widehat{\mathfrak{g} \oplus \mathbb{C}}$ be the Lie algebra $\hat{\mathfrak{g}} \oplus \mathcal{A}$ $(K_{\hat{\mathfrak{g}}} = K_{\mathcal{A}})$, where $\mathcal{A}$ is the Heisenberg algebra with basis $a_n, n \in \mathbb{Z}, K$, and $[a_n, a_m] = n\delta_{n,-m}K$, $[a_n, K] = 0$.

let $\widehat{\mathfrak{g}\oplus\mathbb{C}} = \mathbb{C}d \ltimes \overline{\mathfrak{g}\oplus\mathbb{C}}$ (we add the degree derivation). Let F be the Fock module for $\mathcal{A}$, and $\mathbb{V}_0$ be the basic representation of $\hat{\mathfrak{g}}$, $\mathbb{V}_0 = L_{\omega_0}$. Let $\mathbb{V} = \mathbb{V}_0 \otimes F$ be the basic representation of $\widehat{\mathfrak{g}\oplus\mathbb{C}}$ (we make d act by 0 on the highest weight vector). Given (k,λ), we define a "reductive" subalgebra $\mathfrak{a}_{k,\lambda}$ of $\widehat{\mathfrak{g}\oplus\mathbb{C}}$ as follows.

Let α be a root of $\mathfrak{g}$, and m is an integer with $|m| \le n-1$. Let $N \in \mathbb{Z}_{\geq 0}$. Define the hyperplane

$$H_{\alpha,m,N} = \{(k,\lambda) \mid (\lambda,\alpha) + N + km\} = 0.$$

(note that $(\lambda,\alpha)+N = (\lambda, \alpha+N\delta)$), and $\alpha+N\delta$ is a real root of $\hat{\mathfrak{g}}$).

If $\lambda \in H_{\alpha,m,N}$, then we include $e_{\alpha+m\delta}, e_{-\alpha-m\delta}$ as generators of $\mathfrak{a}_{k,\lambda}$.

Also define the hyperplane

$$E_{m,N} = \{(k,\lambda) \mid km+N=0\}$$ for

$m \in \mathbb{Z}$, $2 \le m \le n$, $N \in \mathbb{Z}$, $\gcd(m,N)=1$.

If $(k,\lambda) \in E_{m,N}$, then we include

$a_{m\ell}, a_{-m\ell}$, $\ell \in \mathbb{Z}\setminus 0$ as generators of $\mathfrak{a}_{k,\lambda}$.

Also, we include $\tilde{\mathfrak{h}}$ into $\mathfrak{a}_{k,\lambda}$ and define $\mathfrak{a}_{k,\lambda}$ as the Lie subalgebra of $\widetilde{\mathfrak{g}\oplus\mathbb{C}}$ generated by all these elements.

The number of f.d. representations of $H_{k,\lambda}$ is now expressed in terms of the structure of the restriction of $\mathbb{V}$ to the subalgebra $\mathfrak{a}_{k,\lambda}$.

Namely, by the standard theory of representations of affine Lie algebras, $\mathbb{V}|_{\mathfrak{a}_{k,\lambda}}$ is a semisimple (in fact, unitary) representation of $\mathfrak{a}_{k,\lambda}$, which has a decomposition

$$\mathbb{V} = \bigoplus_{\mu\in P_+(\mathfrak{a}_{k,\lambda})} L_\mu \otimes \mathrm{Hom}_{\mathfrak{a}_{k,\lambda}}(L_\mu, \mathbb{V}).$$

It's easy to see that μ occuring in this sum satisfy $\mu^2 = -2i$, $i \in \mathbb{Z}_+$.
Indeed, $\mu = \omega_0 + \beta - r\delta$, where $\beta \in Q_{\mathfrak{g}}$, $r \geq 0$, and $r \geq \frac{\beta^2}{2}$. Thus $\mu^2 = -2r + \beta^2$, which is a nonpositive even integer.

Thus the "extremal" case is $\mu^2 = 0$, i.e. $\frac{\beta^2}{2} = r$. If $\mu^2 = 0$, then it's easy to see that $\dim \mathrm{Hom}_{\mathfrak{a}_{k,\lambda}}(L_\mu, \mathbb{V}) = 1$, since $\mathbb{V}[\mu]$ is a 1-dimensional space spanned by an extremal vector w_β in $\mathbb{V}$. So the part of $\mathbb{V}$ corresponding to $\mu^2 = 0$ is

$$\mathbb{V}^{(0)} \cong \bigoplus_{\substack{\mu \in P_+(\mathfrak{a}_{k,\lambda}) \\ \mu^2 = 0}} L_\mu .$$

For $m \geq 1$, let $D_m = \frac{1}{m} \sum_{\ell=1}^{\infty} a_{-m\ell}\, a_{m\ell}$,

<u>Conjecture 5.3</u> 1) If $k \notin \mathbb{Q} \setminus \mathbb{Z}$ then the number of f.d. irred. representations of $H_{k,\lambda}$ is equal to $\dim V^{(0)}[\omega_0 - n\delta]$.

2) If $k \in \mathbb{Q} \setminus \mathbb{Z}$ and the denominator of k is m, then the number of f.d. irred. representations of $H_{k,\lambda}$ is equal to $\dim \mathrm{Ker}\, D_m \big|_{\mathbb{V}^{(0)}[\omega_0 - n\delta]}$.

<u>Exercise 5.1</u> Check that for $n = 1$ this gives the answer explained above.

2) Let $\lambda = \omega_0$, $k \notin \mathbb{Q}$. Check that the number of irreps predicted by the conjecture is

$$\sum_{\substack{\nu \in P_+(\mathfrak{g}) \\ \nu^2 = 2n}} \dim L_\nu[0],$$

where L_ν are irreducible representations of $\mathfrak{g}$.

Conjecture 5.3 implies

<u>Conjecture 5.5</u> The algebra $H_{k,\lambda}$ is simple unless $\lambda \in H_{\alpha,m,N}$ or $\lambda \in E_{m,N}$ for some α, m, N.

Proof of the implication $5.3 \Rightarrow 5.5$ follows from Losev's theory of completions of symplectic reflection algebras.

Conjecture 5.3 and 5.5 are proved by Losev and Bezrukavnikov.

I will explain another conjecture of this type, for cyclotomic case, which was made in the same paper and proved by Shan & Vasserot. It generalizes Conj. 5.3.

Let $\mathcal{O}_{k,\lambda}$ be the category $\mathcal{O}$ of $H_{k,\lambda}$-modules for $T = \mathbb{Z}_\ell$.

We will prove later the following classification theorem for possible supports of objects of $\mathcal{O}_{k,\lambda}$ as $\mathbb{C}[\mathfrak{h}]$-modules.

Let $m \geq 2$. Let $Y_{p,j,m} \subset \mathbb{C}^n$ be the set of points such that some p coordinates are zero, and some j sets of m coordinates consist of equal numbers ($p + jm \leq n$). Also let $Y_p \subset \mathbb{C}^n$ be the set of points where some p coordinates are zero; so $Y_{p,0,m} = Y_p$ $\forall m$.

<u>Proposition 5.6.</u> 1) If $k \notin \mathbb{Q}\setminus\mathbb{Z}$ then the support of every simple object of $\mathcal{O}_{k,\lambda}$ is Y_p for some p.

2) If $k \in \mathbb{Q} - \mathbb{Z}$, then the support of every simple object of $\mathcal{O}_{k,\lambda}$ is $Y_{p,j,m}$ for some p, j, where m is the denominator of k.

Let $K = K_0(\mathcal{O}_{k,\lambda})$, $k \notin \mathbb{Q}\setminus\mathbb{Z}$, and $F_i K$ be the part of K spanned by modules with support in Y_{n-i}: $F_n = K \supset F_{n-1} \supset F_{n-2} \supset \dots \supset F_0$.

Also, if $k \in \mathbb{Q} \setminus \mathbb{Z}$, let $F_{i,j} K$ be the part of K spanned by modules with support in $Y_{n-i-jm, j}$. Let $gr_{\ell} K = F_{\ell} K / F_{\ell+1} K$ in the first case, and $gr_{i,j} K =$

$$= F_{i,j} K / (F_{i-m, j+1} K + F_{i-1,j} K + F_{i,j-1} K).$$

(this makes sense since the union of $Y_{p', j'}$ properly contained in $Y_{p,j}$ is $Y_{p+1,j} \cup Y_{p,j+1} \cup Y_{p+m, j-1}$).

<u>Conjecture</u> 5.7 (Thm of Shan & Vasserot).

1) If $k \notin \mathbb{Q} \setminus \mathbb{Z}$, then

$$gr_{i} K \cong \mathbb{V}^{(i)} [\omega_0 - n\delta],$$

where $\mathbb{V}^{(i)} \overset{def}{=} \bigoplus_{\substack{\mu \in P_+(\mathfrak{a}_{k,\lambda}) \\ \mu^2 = -2i}} L_\mu \otimes \mathrm{Hom}_{\mathfrak{a}_{k,\lambda}} (L_\mu, \mathbb{V})$

2) if $k \in \mathbb{Q} \setminus \mathbb{Z}$ and denominator of k is m then

$$gr_{i,j} K \cong \mathrm{Ker}(D_m - j)\big|_{\mathbb{V}^{(i)}} [\omega_0 - n\delta].$$

Note that in the case of f.d. representations ($j=0$, $i=0$), we get Conjecture 5.3.