Lecture 6. Induction and restriction functors for rational Cherednik algebras

Recall that an important role in the representation theory of S_n is played by the induction and restriction functors induced by the embeddings $S_r \times S_{n-r} \hookrightarrow S_n$. Such functors are also defined for the corresponding finite and affine Hecke algebras (also associated to embeddings), and they play an important role in their representation theory. So we'd like to have such functors for rational Cherednik algebras, which are a degenerate version of double affine Hecke algebra. But here we have a problem — there is no algebra embedding $H_c(S_r) \otimes H_c(S_{n-r})$ into $H_c(S_n)$. Yet, one can define Ind and Res functors in category $\mathcal{O}$, which was done in my paper with Bezrukavnikov. This is done using the fact that the above algebra embedding exists at the level of some completions of these algebras.

Let us now describe the construction of Ind and res functors for rational Cherednik algebras.

Let W be a finite subgroup of $GL(\mathfrak{h})$.

Definition. A parabolic subgroup of W is a subgroup of the form W_b, $b\in\mathfrak{h}$ - stabilizer of the point b. Clearly, $W_b\subset GL(\mathfrak{h}/\mathfrak{h}^{W_b})$.

Let $W'\subset W$ be a parabolic subgroup. We will define induction and restriction functors $\mathrm{Ind}: \mathcal{O}_c(W', \mathfrak{h}/\mathfrak{h}^{W'}) \to \mathcal{O}_c(W,\mathfrak{h})$ and $\mathrm{Res}: \mathcal{O}_c(W,\mathfrak{h}) \to \mathcal{O}(W', \mathfrak{h}/\mathfrak{h}^{W'})$. (where we denote restriction of c to W' also by c) But it turns out that unlike finite and affine Hecke case, these functors have to depend on the choice of $b\in\mathfrak{h}$ such that $W'=W_b$ (and have nontrivial monodromy when b goes around a loop), so we'll denote them Ind_b and Res_b.

Let $H_c(W,\mathfrak{h}) = H_{1,c}(W,\mathfrak{h})$, and define the completion $\widehat{H_c}(W,\mathfrak{h})_0 = \mathbb{C}[[\mathfrak{h}]]\otimes_{\mathbb{C}[\mathfrak{h}]} H_c(W,\mathfrak{h})$ of $H_c(W,\mathfrak{h})$ at the formal neighborhood of 0. This is a formal version of the rational Cherednik algebra, which looks like

$\hat{H}_c(W,\mathfrak{h})_0 = \mathbb{C}[[\mathfrak{h}]] \otimes \mathbb{C}W \otimes S\mathfrak{h}$.

Let $\hat{\mathcal{O}}_c(W,\mathfrak{h})$ be the category of $\hat{H}_c(W,\mathfrak{h})$-modules that are f.generated over $\mathbb{C}[[\mathfrak{h}]]$. We have a completion functor $\hat{\ }: \mathcal{O}_c(W,\mathfrak{h}) \to \hat{\mathcal{O}}_c(W,\mathfrak{h})$,

$$\hat{M} \overset{def}{=} \mathbb{C}[[\mathfrak{h}]] \otimes_{\mathbb{C}[\mathfrak{h}]} M.$$

<u>Lemma 6.1.</u> The functor $\hat{\ }$ is an equivalence of categories. The inverse $E: \hat{\mathcal{O}}_c \to \mathcal{O}_c$ is the functor of $\mathfrak{h}$-nilpotent vectors.

<u>Proof</u>. exercise.

<u>Centralizer construction</u>. Let K be a finite group, $e_K = \frac{1}{|K|}\sum_{x\in K} x$ - symmetrizer. Let $A \supset K$ be an algebra, and $G \supset K$ be another group. Let $Z(G,K,A)$ be the centralizer of A in the right A-module $P = \mathrm{Fun}_K(G,A) = \{f : G \to A,\ f(kg) = kf(g)\ \forall k \in K\}$, i.e. $Z(G,K,A) = \mathrm{End}\, P_A$. Since P is a free right A-module of rank $|G/H|$, we have a non-canonical isomorphism $Z \cong \mathrm{Mat}_{|G/H|}(A)$.

Lemma 6.2. The functor $I: N \mapsto I(N) = P \otimes_A N = \mathrm{Fun}_K(G, N)$ is an equivalence of categories $A\text{-mod} \xrightarrow{\sim} Z\text{-mod}$

Proof. Exercise.

Completions of RCA at arbitrary points. Let $b \in \mathfrak{h}^*$. Let $\mathbb{C}[[\mathfrak{h}]]_{Wb}$ be the completion of $\mathbb{C}[\mathfrak{h}]$ at the orbit Wb of b, i.e. $\mathbb{C}[[\mathfrak{h}]]_{Wb} = \bigoplus_{a \in Wb} \mathbb{C}[[\mathfrak{h}]]_a$.

Define the completion $\widehat{H}_c(W, \mathfrak{h})_b \overset{\text{def}}{=} \mathbb{C}[[\mathfrak{h}]]_{Wb} \otimes_{\mathbb{C}[\mathfrak{h}]} H_c(W, \mathfrak{h})$.

Theorem 6.3. We have a natural isomorphism

$$\theta: \widehat{H}_c(W, \mathfrak{h})_b \to Z(W, W_b, \widehat{H}_c(W_b, \mathfrak{h})_0).$$

This map defines an equivalence of categories $\theta_*: \widehat{H}_c(W, \mathfrak{h})_b\text{-mod} \to Z(W, W_b, \widehat{H}_c(W, \mathfrak{h})_0)\text{-mod}$.

(sketch)
Proof. Recall that $H_c(W, \mathfrak{h})$ is generated by $\mathbb{C}[\mathfrak{h}]$, W, and Dunkl operators

$$D_y = \partial_y - \sum_{s \in S} \frac{2c_s}{1-\lambda_s} (\alpha_s, y) \frac{1}{\alpha_s(x)} (1-s)$$

Now let us complete near the orbit of a point b. So we write $x = wb + (x - wb)$. So we have that $\widehat{H}_c(W, \mathfrak{h})_b$ is generated by $\mathbb{C}[[\mathfrak{h}]]_{Wb}$, W and

$$D_y = \partial_y - \sum_{s\in S} \frac{2c_s}{1-\lambda_s}(\alpha_s, y)\frac{1}{\alpha_s(b') + \alpha_s(x-b')}(1-s),\quad b'\in Wb$$

The terms in the sum are of two types - the ones with $\alpha_s(b')=0$ (singular at $x-b'=0$) and $\alpha_s(b')\neq 0$ (regular at $x-b'=0$)

The regular terms lie in the subalgebra generated by $\mathbb{C}[[\mathfrak{h}]]_{Wb}$ and W.

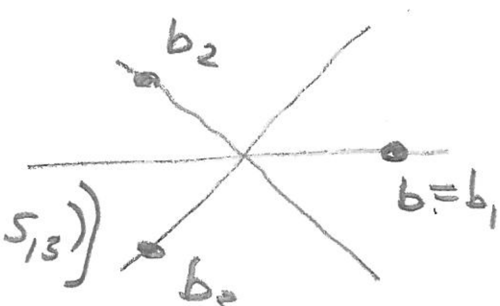
Thus, $\hat{H}_b(W,\mathfrak{h})$ is generated by $\mathbb{C}[[\mathfrak{h}]]_{Wb}$, W, and $\overline{D}_y$, where $\forall b'\in Wb$

$$\overline{D}_y(b') \stackrel{def}{=} \partial_y - \sum_{\substack{s\in S:\\ \alpha_s(b')=0}} \frac{2c_s}{1-\lambda_s}(\alpha_s,y)\frac{1}{\alpha_s(x-b')}(1-s).$$

Here the sum is over the reflections s such that $\alpha_s(b')=b'$, i.e. $sb'=b'$, or $s\in W_{b'}\cong W_b$. Thus the algebra generated by $\mathbb{C}[[\mathfrak{h}]]_{Wb}$, W, and $\overline{D}_y$ should be expressed in some way via $H_c(W_b,\mathfrak{h})$. And indeed, it's easy to show that this algebra

is nothing but $Z(W, W_b, \hat{H}_c(W_b,\mathfrak{h})_0)$.
(to see this, it's enough to note that $\mathbb{C}[[\mathfrak{h}]]_{b'}$, $W_{b'}$, and $\overline{D}_y(b')$ generate a copy of $\hat{H}_c(W_{b'},\mathfrak{h})_0$. ■

Example. $W = S_3$, $\mathfrak{h}$ = reflection repr.,
$b = (1,0)$, $W_b = S_2 = \langle(12)\rangle \in S_3$
$Wb = \{b_1, b_2, b_3\}$ (see picture).



$$D_1 = \partial_1 - c\left(\frac{1}{x_1-x_2}(1-s_{12}) + \frac{1}{x_1-x_3}(1-s_{13})\right)$$

etc.
When complete near b: $\tilde{x} = x - b$

$$D_1 = \partial_1 - c\left(\frac{1}{x_1-x_2}(1-s_{12}) + \frac{1}{x_1-x_3+1}(1-s_{13})\right)$$
$$D_2 = \partial_2 - c\left(\frac{1}{x_2-x_1}(1-s_{21}) + \frac{1}{x_2-x_3+1}(1-s_{23})\right)$$
$$D_3 = \partial_3 - c\left(\frac{1}{x_3-x_1-1}(1-s_{31}) + \frac{1}{x_3-x_2+1}(1-s_{32})\right)$$
$$\Downarrow$$
$$\overline{D}_1(b) = \partial_1 - \frac{c}{x_1-x_2}(1-s_{12})$$
$$\overline{D}_2(b) = \partial_2 - \frac{c}{x_2-x_1}(1-s_{21})$$
$$\overline{D}_3(b) = \partial_3$$

(We use e.g. that $\frac{1}{x_1-x_3+1} = 1 + (x_3-x_1) + (x_3-x_1)^2 + \cdots$)

So the algebra generated by $\mathbb{C}[[\tilde{x}]]$, s_{12} and $\overline{D}_1, \overline{D}_2, \overline{D}_3$ is $\hat{H}_c(S_2,\mathfrak{h})_0$.

Definition. $\hat{\mathcal{O}}_c(W,\mathfrak{h})^b$ is the category of $\hat{H}_c(W,\mathfrak{h})_b$-modules which are finitely generated over $\mathbb{C}[[\mathfrak{h}]]_b$.

We have the completion functor
$\hat{}_b : \mathcal{O}_c(W,\mathfrak{h}) \to \hat{\mathcal{O}}_c(W,\mathfrak{h})^b$, $M \mapsto \hat{M}_b$,
$\hat{M}_b \overset{def}{=} \mathbb{C}[[\mathfrak{h}]]_{Wb} \otimes_{\mathbb{C}[\mathfrak{h}]} M$.

Also we have a functor in the opposite direction: $E^b : \hat{\mathcal{O}}_c(W,\mathfrak{h})^b \to \mathcal{O}_c(W,\mathfrak{h})$,
$E^b(M) = \{\mathfrak{h}\text{-nilpotent vectors in } M\}$.
(it can be shown that $E^b(M)$ is finitely generated over $\mathbb{C}[\mathfrak{h}]$).

Theorem. 3.4. The functors E^b, $\hat{}_b$ are exact, and E^b is right adjoint to $\hat{}_b$.

Proof. $\hat{}_b$ is exact because the completion functor for $\mathbb{C}[[\mathfrak{h}]]$-modules is exact ($\mathbb{C}[[\mathfrak{h}]]_b$ is a flat $\mathbb{C}[\mathfrak{h}]$-module).
$Hom(\hat{M}_b, N)$ maps to $Hom(M,N)$ by pullback through the map $M \to \hat{M}_b$.
The induced map $\xi : Hom(\hat{M}_b, N) \to Hom(M,N)$ is surjective, because any morphism from M to

an object of $\hat{\mathcal{O}}_c(W,\mathfrak{h})^b$ automatically factors through $\hat{M}_b$. Also, ξ is injective since the image of M is dense in $\hat{M}_b$. So $\xi : \mathrm{Hom}(\hat{M}_b, N) \xrightarrow{\sim} \mathrm{Hom}(M,N)$ is an isomorphism. Also, it's clear that any $\alpha : M \to N$ actually lands in $E^b(N)$, as M is $\mathfrak{h}$-nilpotent.

To show that E^b is exact, note that $E^b(N) = (\widehat{N^*})^*$, where $*$ denotes the continuous dual with respect to the adic topology, and $\wedge$ denotes the completion at o as before. To check this is an exercise. Exactness now follows from the fact that the completion functor is exact. $\square$

Remark. Note that $\hat{}_0 = \hat{}$, $E^0 = E$, and in this case they are inverse.

Finally, we will need the equivalence $\zeta : \mathcal{O}_c(W,\mathfrak{h}) \xrightarrow{\sim} \mathcal{O}_c(W, \mathfrak{h}/\mathfrak{h}^W)$, which is constructed in an obvious way.

Now let $W' \subset W$ be a parabolic, and $b \in \mathfrak{h}$ such that $W_b = W'$.

Now we define the functors Ind_b, Res_b by commutative diagram:

$$\begin{tikzcd}
 & \mathcal{O}_c(W,\mathfrak{h}) \arrow[dr, "\wedge_b"] \arrow[ddl, bend left, "\mathrm{Res}_b"] & \\
 & & \hat{\mathcal{O}}_c(W,\mathfrak{h})^b \arrow[ul, bend right, "E^b"'] \arrow[d, "\wr"', "I^{-1}\circ\theta_*"] \\
\mathcal{O}_c(W',\mathfrak{h}/\mathfrak{h}^{W'}) \arrow[uur, bend left, "\mathrm{Ind}_b"] \arrow[dr, bend left, "\zeta^{-1}", "\sim"'] & & \hat{\mathcal{O}}_c(W',\mathfrak{h}) \arrow[u, bend right, "\theta_*^{-1}\circ I"', "\wr"] \arrow[dl, "E", "\sim"'] \\
 & \mathcal{O}_c(W',\mathfrak{h}) \arrow[ul, "\zeta", "\sim"'] \arrow[ur, bend right, "\wedge"', "\sim"] &
\end{tikzcd}$$

<u>Thm 6.5.</u> These functors are exact, and Ind_b is right adjoint to Res_b.

<u>Pf.</u> Follows from the above.

<u>Theorem 6.6.</u> Suppose c is generic (i.e. $\mathcal{O}_c(W,\mathfrak{h})$, $\mathcal{O}_c(W',\mathfrak{h}/\mathfrak{h}^{W'})$ are semis.). Then, upon identification $\mathcal{O}_c(W,\mathfrak{h}) \simeq \mathrm{Rep}\,W$, and similarly for W' (using induction functor) the functors Ind_b and Res_b get identified with the usual functors $\mathrm{Ind}_{W'}^{W}$, $\mathrm{Res}_{W'}^{W}$. Moreover, they always act as $\mathrm{Ind}_{W'}^{W}$ and $\mathrm{Res}_{W'}^{W}$ at the level of Grothendieck groups.

Proof. It is easy to see that the statement holds for $c=0$ (i.e, for equivariant D-modules). Also, the functors Ind_b and Res_b are a flat family of functors with respect to c (since they are defined using completion), so the rest of the theorem follows.

Corollary 6.7 (of Th. 3.5).
Res_b maps projectives to projectives, Ind_b injectives to injectives.

Thm 6.8 (Losev) Ind_b is isomorphic to the left adjoint of Res_b. Thus, Res_b maps injectives to injectives and Ind_b maps projectives to projectives.

This theorem is hard and we will not give a proof.

Thus, the functors Ind_b and Res_b are biadjoint.

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Let us now show that induction and restriction functors are compatible with supports.

Given $M \in \mathcal{O}_c(W, \mathfrak{h})$, its support $\mathrm{Supp}(M) \subseteq \mathfrak{h}$ as a $\mathbb{C}[\mathfrak{h}]$-module is a subvariety of $\mathfrak{h}$.

Dependence of Res_b and Ind_b on b.

For a parabolic subgroup $W' \subset W$, let $\mathfrak{h}^{W'}_{reg} \subset \mathfrak{h}^{W}$ be the open subset of points whose stabilizer is exactly W'. If W' is fixed, the functors Res_b, Ind_b are thus functions of b.

Theorem 6.9. The functors Res_b, Ind_b form local systems of functors on $\mathfrak{h}^{W'}_{reg}$. In other words, if we denote by $\mathrm{Loc}(\mathfrak{h}^{W'}_{reg})$ the category of local systems on $\mathfrak{h}^{W'}_{reg}$ with regular singularities, then we have functors

$$\mathrm{Res}: \mathcal{O}_c(W,\mathfrak{h}) \to \mathcal{O}_c(W', \mathfrak{h}/\mathfrak{h}^{W'}) \boxtimes \mathrm{Loc}(\mathfrak{h}^{W'}_{reg}),$$

$$\mathrm{Ind}: \mathcal{O}_c(W', \mathfrak{h}/\mathfrak{h}^{W'}) \to \mathcal{O}_c(W,\mathfrak{h}) \boxtimes \mathrm{Loc}(\mathfrak{h}^{W'}_{reg})$$

such that Res_b, Ind_b are the fibers of Res, Ind at b. In other words, we have a flat connection with respect to b on Res_b, Ind_b.

Corollary 6.10. Res_b, Ind_b don't depend on b up to an isomorphism.

Corollary follows directly from the theorem.

Proof of Theorem 6.9. Let $\widehat{H}_c(W,\mathfrak{h})_{\mathfrak{h}^{W'}_{reg}}$ be the completion of $\widehat{H}_c(W,\mathfrak{h})$ near $\mathfrak{h}^{W'}_{reg}$ (as a sheaf on $\mathfrak{h}^{W'}_{reg}$)

Similarly to Theorem 6.3, we have an isomorphism

$$\theta: \widehat{H}_c(W,\mathfrak{h})_{\mathfrak{h}^{W'}_{reg}} \xrightarrow{\sim} Z(W,W',\widehat{H}_c(W',\mathfrak{h}/\mathfrak{h}^{W'})_0)\widehat{\otimes} D(\mathfrak{h}^{W'}_{reg})$$

where $D(\mathfrak{h}^{W'}_{reg})$ is the sheaf of diff. operators of $\mathfrak{h}^{W'}_{reg}$. Now, repeating the construction of Res_b, Ind_b, we can construct the functors Res, Ind with the required properties. E.g.

$$Res(M) = (E \circ I^{-1} \circ \theta_*)(\widehat{M}_{\mathfrak{h}^{W'}_{reg}}).$$

<u>Example</u>. Suppose c is generic. Then Res_b, Ind_b are same as for groups.

So $Res_b(\Delta_c(\tau)) \cong \bigoplus_{\sigma \in Irrep W'} \Delta_c(\sigma) \otimes Hom_{W'}(\sigma,\tau)$.

Thus, we are suppose to get a flat connection on $\mathfrak{h}^{W'}_{reg}$ with fiber $Hom_{W'}(\sigma,\tau)$, $\forall \tau \in Irrep W$, $\sigma \in Irrep W'$.

<u>Proposition 8.11</u> This connection has the connection form

$$\sum_{s \in S, s \notin W'} \frac{2c_s}{1-\lambda_s}\frac{d\alpha_s}{\alpha_s}(s-1)$$

<u>Proof</u>. Direct computation. The answer is not surprising, since these are exactly the "regular" terms that we remove in Th 6.3, 6.9. We'll talk more about it later.

Proposition 6.12. $b \in \mathrm{Supp}(M) \iff \mathrm{Res}_b(M) \neq 0$.

Proof. Straightforward from the definition of Res_b.

Corollary 6.13 Consider the stratification of $\mathfrak{h}$ by the sets $\mathfrak{h}^{W'}_{reg}$, where W' runs over the set $\mathrm{Par}(W)$ of parabolic subgroups of W. Then $\forall M \in \mathcal{O}_c(W,\mathfrak{h})$, $\mathrm{Supp}(M)$ is a union of strata of this stratification. (Thus, we get the classification of supports from Lecture 5).

Pf. Follows from Cor 6.10 and Prop 6.12.

Proposition 6.14. The support of $\mathrm{Res}_b(M)$ is the union of all $\left(\mathfrak{h}/\mathfrak{h}^{W'}\right)^{W''}$ for parabolics $W'' \subset W'$ such that $\mathfrak{h}^{W''}$ is contained in $\mathrm{Supp}(M)$.

Proof. If $\mathrm{Supp}M \supset \mathfrak{h}^{W''} \not\ni b$ then $\mathfrak{h}^{W''}$ does not contribute to the support of $\mathrm{Res}_b(M)$, and if $\mathfrak{h}^{W''} \ni b$ then it does. This implies the statement.

Proposition 6.15. The support of $\mathrm{Ind}_b(N)$ is the union of W-translates of all $\mathfrak{h}^{W''}$ for $W'' \subset W'$ and $\left(\mathfrak{h}/\mathfrak{h}^{W'}\right)^{W''}$ contained in the support of N.

Proof. Similar to the proof of Prop 6.14.

Thus, e.g. the support of $\mathrm{Res}_b(M)$ is simply obtained by taking formal neighborhood of b in the in Supp M, and then modding out by $\mathfrak{h}^{W'}$ and taking the corresponding conical variety.

<u>Proposition 6.16</u> (Ginzburg). If $M \in \mathcal{O}_c(W, \mathfrak{h})$ is simple then $\mathrm{Supp}\, M/W$ is irreducible.

<u>Proof</u>. Exercise.

<u>Proposition 6.17</u>. Let $W' \subset W$ be a parabolic. Then $W \cdot \mathfrak{h}^{W'}$ is $\mathrm{Supp}(M)$ for some $M \in \mathcal{O}_c(W, \mathfrak{h})$ if and only if $H_c(W', \mathfrak{h}/\mathfrak{h}^{W'})$ admits a finite dimensional representation.

<u>Pf.</u> $\Rightarrow$ Take $\mathrm{Res}_b(M)$, $b \in \mathfrak{h}^{W'}_{reg}$. Then $\mathrm{Supp}(\mathrm{Res}_b(M)) = \{0\}$ by Prop 6.14, and $\mathrm{Res}_b(M) \neq 0$ by Prop 6.12, so $\mathrm{Res}_b(M)$ is a f.d. representation.
$\Leftarrow$ If N is a f.d. repr. of $H_c(W', \mathfrak{h}/\mathfrak{h}^{W'})$, set $M = \mathrm{Ind}_b(N)$, $b \in \mathfrak{h}^{W'}_{reg}$. Then by Prop 6.15, $\mathrm{Supp}\, M = W \cdot \mathfrak{h}^{W'}$.

<u>Corollary 6.18</u> If $W = S_n$, $\mathfrak{h} = \mathbb{C}^n$, $c \in \mathbb{Q}$, then the supports of simple modules are $X_{i,n}$, which are unions of S_n-translates of the subspaces in $\mathbb{C}^n$ defined by $x_1 = \dots = x_m$, $x_{m+1} = \dots = x_{2m}$, $x_{(i-1)m+1} = \dots = x_{im}$, where m is the denominator of c.

<u>Proof</u>. This follows from the classification

of finite dimensional representations of $H_c(S_n)$.

<u>Theorem 6.19</u> (Losev). $\mathrm{Res}_b(M^+) \cong \mathrm{Res}_b(M)^+$, $\mathrm{Ind}_b(M^+) = \mathrm{Ind}_b(M^+)$.

<u>Proof</u>. This is hard and we will not give a proof. It implies the biadjointness as a corollary.

<u>Theorem 6.20</u>. Res_b and Ind_b map standard objects to standardly filtered objects, hence preserve the category of standardly filtered objects.

<u>Proof</u>. It is easy to see by definition that Res_b maps $\mathbb{C}[\mathfrak{h}]$-free objects to $\mathbb{C}[\mathfrak{h}/\mathfrak{h}^W]$-free ones, and similarly for Ind_b. On the other hand, we have the following lemma, which implies the theorem.

<u>Lemma 6.21</u>. $M \in \mathcal{O}_c$ is standardly filtered if and only if M is free as a $\mathbb{C}[\mathfrak{h}]$-module.

<u>Pf.</u> We only need to prove the "if" direction. Let $M \in \mathcal{O}_c$ be $\mathbb{C}[\mathfrak{h}]$-free. We have

$$\mathrm{Ext}^1_{H_c}(M, \nabla(\tau)) \overset{\text{Shapiro}}{=} \mathrm{Ext}^1_{\mathbb{C}W \ltimes S\mathfrak{h}}(M, \tau) = 0$$

since M is free. Thus M is standardly filtered by the theory of highest wt categories