Preamble to lecture 8.

In this lecture we will use the fact that the localization of the rational Cherednik algebra $H_c(W,\mathfrak{h})$ to the complement of the reflection hyperplanes is the smash product $\mathcal{D}(\mathfrak{h}_{reg})\# W$, so the localization of a Cherednik algebra module is an equivariant $\mathcal{D}$-module ~~over~~ on $\mathfrak{h}_{reg}$. This will allow us to find out how many modules with maximal support we have, which relates to conjecture from lecture 5.

Let $W \subset GL(\mathfrak{h})$ be a complex refl. group.

Lecture 8. The KZ functor.

Recall that the Dunkl representation defines an isomorphism $\eta: H_c(W,\mathfrak{h})_{loc} \xrightarrow{\sim} \mathcal{D}(\mathfrak{h}_{reg}) \# W$, where $H_c(W,\mathfrak{h})_{loc} \overset{def}{=} H_c(W,\mathfrak{h})[\delta^{-1}]$, and $\delta = \prod \alpha_s$. Thus we get the localization functor $\mathcal{O}_c(W,\mathfrak{h}) \longrightarrow \mathcal{D}(\mathfrak{h}_{reg}) \# W\text{-mod}$ given by $M \mapsto \mathbb{C}[\mathfrak{h}][\delta^{-1}] \otimes_{\mathbb{C}[\mathfrak{h}]} M$, from $\mathcal{O}_c$ to the category of W-equivariant $\mathcal{D}(\mathfrak{h}_{reg})$-modules. We will denote this functor KZ for "Knizhnik-Zamolodchikov", since as we will see, it is related to the Knizhnik-Zamolodchikov equations.

<u>Theorem 8.1.</u> For $M \in \mathcal{O}_c(W,\mathfrak{h})$, $KZ(M)$ is a local system with regular singularities on $\mathfrak{h}_{reg}/W$ (i.e. a W-equivariant local system on $\mathfrak{h}_{reg}$).

<u>Proof.</u> By definition, $KZ(M)$ is $\mathcal{O}$-coherent, hence it is a W-equivariant local system on $\mathfrak{h}_{reg}$, i.e. a local system on $\mathfrak{h}_{reg}/W$. We will show below that $KZ(\Delta(\tau))$ is the KZ connection, so it has regular singularities. Hence $KZ(P(\tau))$ has regular singularities (as $P(\tau)$ has a standard filtration), thus $KZ(M)$ for all M has regular singularities

<u>Theorem 8.2.</u> $KZ(\Delta(\tau))$ is the D-module corresponding to the KZ connection, i.e. the connection on the trivial bundle over $\mathfrak{h}_{reg}$ with fiber τ, and with connection form
$$\omega = \sum_{s\in S} \frac{2c_s}{1-\lambda_s} \frac{d\alpha_s}{\alpha_s}(1-\rho_\tau(s)).$$

<u>Proof.</u> Recall that $y\in\mathfrak{h}$ acts on $\Delta(\tau)=\mathbb{C}[\mathfrak{h}]\otimes\tau$ by Dunkl operators
$$D_y = \partial_y + \sum_{s\in S}\frac{2c_s}{1-\lambda_s}\frac{\alpha_s(y)}{\alpha_s}(s-1)\otimes\rho_\tau(s)$$
So the element $\frac{\partial}{\partial y}\in \mathcal{D}(\mathfrak{h}_{reg})\# W$ acts on $f\otimes v$ (for $f\in \mathbb{C}[\mathfrak{h}][\delta^{-1}]$, $v\in\tau$) by
$$\frac{\partial}{\partial y}(f\otimes v) = \left(D_y + \sum_{s\in S}\frac{2c_s}{1-\lambda_s}\frac{\alpha_s(y)}{\alpha_s}(1\otimes 1 - s\otimes s)\right)(f\otimes v)$$
$$= \sum_{s\in S}\frac{2c_s}{1-\lambda_s}\frac{\alpha_s(y)}{\alpha_s} f\otimes(1-\rho_\tau(s))v.$$
This implies the statement $\blacksquare$

<u>Example.</u> Let $W=S_n$. Then
$$\omega = c\sum_{i<j}\frac{d(z_i-z_j)}{z_i-z_j}(1-s_{ij})|_\tau,$$
which is the KZ connection form for S_n.

<u>Remark.</u> Note that since W is a CRG, any parabolic contains reflections, so W acts freely on $\mathfrak{h}_{reg}$.

<u>Theorem 8.3.</u> The functor KZ is exact, and its kernel is the subcategory $\mathcal{O}_c^{tor}(W,\mathfrak{h})$ of objects whose support is a proper subvariety of $\mathfrak{h}$.

<u>Proof.</u> The first statement is obvious, as $\mathbb{C}[\mathfrak{h}][\delta^{-1}]$ is flat over $\mathbb{C}[\mathfrak{h}]$. By results of lecture 6, $\forall M \in \mathcal{O}_c$, Supp M is a union of strata $\mathfrak{h}^{W'}_{reg}$, $W' \subset W$ a parabolic. But by Chevalley theorem, W' is itself a complex reflection group, so $\underbrace{\mathfrak{h}^{W'} \subseteq \{\delta = 0\}}_{\text{if } W' \neq W}$. Hence either Supp $M = \mathfrak{h}$ or Supp $M \subset \{\delta = 0\}$. It's clear that $KZ(M) \neq 0$ in the first case and $KZ(M) = 0$ in the second case. Thus, theorem follows.

<u>Corollary 8.4.</u> KZ descends to a functor $\overline{KZ}: \mathcal{O}_c(W,\mathfrak{h}) / \mathcal{O}_c^{tor}(W,\mathfrak{h}) \to \mathcal{D}(\mathfrak{h}_{reg})\#W\text{-mod}$ with trivial kernel.

Rem. Note that KZ is a special case of the restriction functor Res (local system valued version), for parabolic $W' = \{1\}$.

<u>Hecke algebras.</u> Since $KZ(M)$ is a local system with regular singularities, it makes sense

to consider the corresponding representation of the fundamental group, attached to this local system by the Riemann–Hilbert correspondence. The fundamental group $\pi_1(\mathfrak{h}_{reg}/W)$ is called the braid group of W, and denoted by B_W. If we glue back the locus $\delta=0$ (i.e. the union of reflection hyperplanes) mod W, we will get the orbifold $\mathfrak{h}/W$. As $\pi_1^{orb}(\mathfrak{h}/W)=W$, by van Kampen's theorem, we have a surjective homomorphism $\pi: B_W \to W$, and $\operatorname{Ker}\pi$ is generated, as a normal subgroup, by the relations $\gamma_E=1$, where γ_E is the path around the hyperplane $E=\mathfrak{h}^s$, $s\in S$. Let $T_E\in B_W$ be the path around the image of $\mathfrak{h}^s$ in $\mathfrak{h}_{reg}/W$ (defined up to conjugacy). Then $\gamma_E = T_E^{n_E}$, where n_E is the order of $\operatorname{Stab}(x)$, $x\in E$ generic. Thus,

$$W \cong B_W / \langle T_E^{n_E}=1 \rangle$$

where E runs over the set of reflection hyperplanes.

We will now define a deformation of $\mathbb{C}W$ which is called the Hecke algebra of W. It's defined by Broue, Malle, and Rouquier.

Definition. The Hecke algebra of W, $\mathcal{H}_q(W)$, is the algebra

$$\mathcal{H}_q(W) = \mathbb{C}B_W \Big/ \left\langle (T_E - 1) \prod_{m=1}^{n_E-1} \left(T_E - e^{\frac{2\pi i m}{n_E}} q_m(E)\right) \right\rangle \quad (*)$$

where $q_m(E)$ are conjugation invariant, and E runs over all the reflection hyperplanes. E.g. $\mathcal{H}_1(W) = \mathbb{C}W$.

<u>Example</u>. If W is a real reflection group then $n_E = 2$, so we have just one parameter $q(E)$ for each E, and

$$\mathcal{H}_q(W) = \mathbb{C}B_W / \langle (T_E - 1)(T_E + q(E)) \rangle \quad \text{4a}$$

<u>The monodromy functor</u> Now consider the KZ connection $D = d + \omega$, $\omega = \sum_{s \in S} \frac{2c_s}{1-\lambda_s} \frac{d\alpha_s}{\alpha_s}(1 - \rho_c(s))$. let T_E be the monodromy operator around E in $\mathfrak{h}_{reg}/W$ for this connection. let us calculate the eigenvalues of T_E. Let $s_E \in W$ be the element acting trivially on E and by $\lambda_E = \exp\left(\frac{2\pi i}{n_E}\right)$ on the normal line. Let $q_m(E) = \exp\left(+\frac{2\pi i}{n_E} \sum_{j=1}^{n_E-1} \frac{2c_{s_E^j}(1-\lambda_E^{jm})}{1-\lambda_E^j}\right)$.
Then from the above 1-variable calculation we get that for generic c there is a monodromy functor

The monodromy of KZ connection (1 variable).

Consider the KZ connection in 1 variable,

$\nabla = d + \omega$, $\quad \omega = +\sum_{j=1}^{n-1} k_j \frac{dz}{z}(1-s^j)$, where $sz = \lambda z$,

λ - a primitive n-th root of 1.

If τ_m is a character of $W = \langle 1, s, s^2, .., s^{n-1} \rangle$ given by $\tau_m(s) = \lambda^m$ then the KZ connection with values in τ_m has the form

$$d + \sum_{j=1}^{n-1} k_j (1-\lambda^{jm}) \frac{dz}{z},$$

so the flat section is $z^{-\sum_{j=1}^{n-1} k_j (1-\lambda^{jm})}$.

We regard this system as W-equivariant system, which gives a local system on $\mathbb{C}^*/W \cong \mathbb{C}^*$ (via $z \to z^n$). The monodromy of the latter is $\lambda^m q_m$

$q_m \stackrel{\text{def}}{=} \exp\left(\frac{2\pi i}{n} \sum_{j=1}^{n-1} k_j (1-\lambda^{jm})\right)$. Thus, setting $k_j = \frac{+2c_{s^j}}{1-\lambda^j}$, we get that (for generic c) there is a monodromy functor

$KZ: \mathcal{O}_c(W, \mathbb{C}) \to Rep\, \mathcal{H}_q(W)$, $q = (q_1, .., q_{n-1})$.

Indeed

$KZ: \mathcal{O}_c(W,\mathfrak{h}) \to \mathrm{Rep}\,\mathcal{H}_q(W)$.

<u>Proposition</u> 8.5. The functor KZ in fact lands in $\mathrm{Rep}\,\mathcal{H}_q(W)$ for all c (not just for generic c).

<u>Proof.</u> We need to show that $\forall M \in \mathcal{O}_c$, the monodromy representation of $KZ(M)$, which is a representation of the braid group B_W, in fact factors through the Hecke algebra $\mathcal{H}_q(W)$. For this purpose, we need to show that the monodromy operators T_E in $KZ(M)$ satisfy the polynomial equation $(*)$. It suffices to show this for projectives, since any $M \in \mathcal{O}_c$ has a projective cover, and the functor KZ is exact. But projectives admit a flat deformation to generic c, so the statement is valid by a limiting argument.

<u>Corollary 8.6.</u> The formal deformation of $\mathcal{H}_1(W) = \mathbb{C}W$ into $\mathcal{H}_q(W)$ is flat.

<u>Proof.</u> Given $\tau \in \mathrm{Irrep}\,W$, we have the representation τ_q of $\mathcal{H}_q(W)$, which is the monodromy representation of $KZ(\Delta(\tau))$. Clearly, τ_q is a flat deformation of τ. So any repr of $\mathcal{H}_1(W) = \mathbb{C}W$ admits a flat deformation to a repr. of $\mathcal{H}_q(W)$, hence $\mathcal{H}_q(W)$ is formally flat.

<u>Conjecture 8.6.</u> (Broué, Malle, Rouquier). $\mathcal{H}_q(W)$, as an algebra over $\mathbb{C}[q,q^{-1}]$, is a free module over $\mathbb{C}[q,q^{-1}]$ of rank $|W|$. In particular, <u>for numerical</u> q, $\dim \mathcal{H}_q(W)=|W|$. This conjecture is known for many complex reflection groups, e.g. for Coxeter groups it follows from the theory of length. Also it holds for the groups $G(\ell,k,n)$. But for a lot of groups it is open, including some rank <u>2 groups.</u>

<u>Example</u>: The Hecke algebra of type A. Let $W=S_n$, $\mathfrak{h}=\mathbb{C}^n$. Then the braid group $B_W=B_n$ is generated by T_i, $i=1,..,n-1$, with relations $T_iT_j=T_jT_i$ for $|i-j|\geq 2$, $T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1}$. Namely, T_i are the half-circles around $z_i=z_{i+1}$. In B_W, all T_E are conjugate to T_i for any i, so $q(E)$ is a single parameter $q=e^{2\pi i c}$, and we have

$$\mathcal{H}_q(S_n)=\mathbb{C}B_n\Big/(T_i-1)(T_i+q)=0 \quad \forall i.$$

This algebra has a basis T_w, $w\in S_n$, where

$T_w = T_{s_1} \cdots T_{s_r}$ for any reduced expression $w = s_1 \cdots s_r$ (it does not depend on the choice of the reduced expression).

<u>Example</u>. $W = G(\ell, 1, n)$. $\mathfrak{h} = \mathbb{C}^n$. Then the reflection hyperplanes are $z_i = \lambda^m z_j$, $\lambda = e^{\frac{2\pi i}{\ell}}$, $m = 0, \dots, \ell-1$, and $z_i = 0$.

The braid group B_W of $\mathfrak{h}_{reg}/W$ has the following structure. Let $x_i = z_i^m$, then $\mathfrak{h}_{reg}/W$ is identified with $\{(x_1, \dots, x_n) \mid {x_i \neq x_j \atop x_i \neq 0}\}/S_n$, the "affine" braid group, $\pi^{-1}(S_n)$, where $\pi : B_{n+1} \to S_{n+1}$.

This group is generated by T_0^2 and T_i, $i = 1, \dots, n-1$. and since $T_0 T_1 T_0 = T_1 T_0 T_1$, we have

$$T_0^2 T_1 T_0^2 T_1 = T_0 \underbrace{T_1 T_0 T_1} T_0 T_1 = \underbrace{T_1 T_0 T_1} \underbrace{T_0 T_1 T_0}$$

$$= T_1 T_0^2 T_1 T_0^2.$$

Thus setting T to be T_0^2, we get the usual type A relations for T_i and also $T T_i T T_i = T_i T T_i T$. One can show that these relations are in fact defining (exercise).

Now consider the Hecke algebra $\mathcal{H}_q(W)$. We have two conjugacy classes of reflection hyperplanes, $z_i = z_j$ and $z_i = 0$.

to the hyperplane $z_i = z_j$ attached is just one parameter $q = e^{2\pi i c}$. To the hyperplane $z_j = 0$, attached are $\ell-1$ parameters $q_j, j = 1, .., \ell-1$, and the Hecke algebra is defined as follows:

$$\mathcal{H}_{(q, q_1, .., q_{\ell-1})}(W) =$$

$$\mathbb{C}B_W / (T_i - 1)(T_i + q) = 0$$
$$(T-1)(T-\lambda q_1)\cdots(T-\lambda^{\ell-1} q_{\ell-1}) = 0.$$

We have seen that the functor

$$KZ : \mathcal{O}_c(W, \mathfrak{h}) \longrightarrow \operatorname{Rep} \mathcal{H}_q(W)$$

is exact. Thus, it is represented by a projective $P = P_{KZ} \in \mathcal{O}_c(W, \mathfrak{h})$, which is equipped with an algebra morphism

$$\varphi : \mathcal{H}_q(W) \longrightarrow \operatorname{End}_{\mathcal{O}}(P_{KZ})^{opp}.$$

It is easy to see that $P_{KZ} = \bigoplus_{\tau \in \operatorname{Irr} W} d_\tau P(\tau)$,

~~Indeed~~ where: $\dim \sigma = \sum [\Delta(\sigma) : L(\tau)] d_\tau$.

Indeed, $\dim \operatorname{Hom}(P_{KZ}, \Delta(\sigma)) = \dim \sigma = \sum [\Delta(\sigma) : L(\tau)] d_\tau$

Also note that $P_{KZ} = \operatorname{Ind}(\mathbb{C})$ for the parabolic subgroup $1 \in W$. Indeed,

<u>Lemma 8.7.</u> $\mathcal{KZ} : \mathcal{O}_c/\mathcal{O}_c^{tor} \to \mathcal{H}_q(W)\text{-mod}$ is a fully faithful exact functor whose image is closed under taking subobjects and quotients.

<u>Proof.</u>

$$\begin{tikzcd}
 & \mathcal{D}(\mathfrak{h}_{reg})\#W\text{-mod}_{RS}^{\mathcal{O}\text{-coh}} \arrow[d, phantom, "\cong \ \text{RH corr. (Deligne)}"] \\
\mathcal{O}_c/\mathcal{O}_c^{tor} \arrow[ur, "KZ"] \arrow[dr, dashed, "\mathcal{KZ}"'] & \mathbb{C}B_W\text{-mod} \\
 & \mathcal{H}_q(W)\text{-mod} \arrow[u, hook, "\text{full subcat}"']
\end{tikzcd}$$

So it's enough to show that the functor $KZ : \mathcal{O}_c/\mathcal{O}_c^{tor} \to \mathcal{D}(\mathfrak{h}_{reg})\#W\text{-mod}$ is exact fully faithful, with image closed under subs and quotients. But this follows from the fact that

$\mathcal{D}(\mathfrak{h}_{reg})\#W$ is a localization of H_c.

$\mathrm{Ind}(\mathbb{C})$ is Projective (since Ind maps projectives to projectives), and

$$\mathrm{Hom}(\mathrm{Ind}(\mathbb{C}), \Delta(\tau)) = \mathrm{Hom}(\mathbb{C}, \mathrm{Res}\,\Delta(\tau)) \cong \tau.$$

Also, P_{KZ} is injective (as Ind maps injectives to injectives). ← 9A

Proposition 8.7. If $\dim \mathcal{H}_q(W) = |W|$ then the map $\phi: \mathcal{H}_q(W) \to \mathrm{End}(P_{KZ})^{op}$ is an isomorphism. Thus, the functor $KZ: \mathcal{O}_c/\mathcal{O}_c^{tor} \to \mathrm{Rep}\,\mathcal{H}_q(W)$ is an equivalence of categories.

Pf. Let's check equality of dimensions.

$$\dim \mathrm{End}_{\mathcal{O}}(P_{KZ}) = \sum_{\sigma,\tau} d_\sigma d_\tau \dim \mathrm{Hom}(P(\sigma), P(\tau))$$

$$= \sum_{\sigma,\tau} d_\sigma d_\tau [P(\tau):\sigma] = \sum_{\sigma,\tau,\lambda} d_\sigma d_\tau [P(\tau):\Delta(\lambda)]\cdot[\Delta(\lambda):\sigma]$$

$$= \sum_{\sigma,\tau,\lambda} d_\sigma d_\tau \underbrace{[\Delta(\lambda):\tau]}_{\text{BGG duality}} [\Delta(\lambda):\sigma] = \sum_\lambda (\dim\lambda)^2 = |W|.$$

It remains to show that ϕ is ~~injective~~ surjective.

~~i.e. any f.d. $\mathcal{H}_q(W)$-module E is of the form $KZ(M)$ for some M. To show this, we use the Riemann-Hilbert correspondence. There exists a local system with regular singularities whose monodromy~~

But by Lemma 8.7, if $\overline{P}_{KZ}$ is the image of P_{KZ} under $\mathcal{O}_c \to \mathcal{O}_c/\mathcal{O}_c^{tor}$, then the map $\mathcal{H}_q(W) \to \mathrm{End}(\overline{P}_{KZ})^{opp}$ is surjective (as it corresponds to the inclusion $\mathcal{O}_c/\mathcal{O}_c^{tor} \hookrightarrow \mathrm{Rep}\,\mathcal{H}_q(W)$). But the map $\mathrm{End}\,P_{KZ} \to \mathrm{End}\,\overline{P}_{KZ}$ is an isomorphism.

~~as a P_{KZ} does not~~

Indeed, $\mathrm{Hom}(\overline{P}_{KZ}, \overline{P}_{KZ}) =$

$$= \varinjlim_{P_{KZ}/X,\ Y \in \mathcal{O}_c^{tor}} \mathrm{Hom}(X, P_{KZ}/Y).$$

But $X = 0$ since all simple quotients of P_{KZ} are not in $\mathcal{O}_c^{tor}$ ($\mathrm{Hom}(P_{KZ}, L(\tau)) = 0$ if $L(\tau) \in \mathcal{O}_c^{tor}$) [$\overset{KZ(L(\tau))}{=}$]

So $\mathrm{Hom}(\overline{P}_{KZ}, \overline{P}_{KZ}) = \varinjlim_{Y \in \mathcal{O}_c^{tor}} \mathrm{Hom}(P_{KZ}, P_{KZ}/Y)$.

But $\mathrm{Hom}(P_{KZ}, Y) = 0$ for $Y \in \mathcal{O}_c^{tor}$, so this is the same as $\mathrm{Hom}(P_{KZ}, P_{KZ})$.

Thus, ϕ is an isomorphism. ∎

Lemma 8.9. Any standardly filtered (in particular, projective) $M \in \mathcal{O}_c$ is a subobject of $P_{KZ}^{\oplus N}$.

Proof. We have an adjunction morphism $M \to \mathrm{Ind}_1^W \mathrm{Res}_1^W M$. This morphism is injective, since it's injective upon localization, and M is torsion-free. But $\mathrm{Ind}\,\mathrm{Res}\, M$ is a multiple of $\mathrm{Ind}\,\mathbb{C}$, which is P_{KZ}.

Lemma 8.10. If M is projective then $M \hookrightarrow P_{KZ}^{\oplus N}$ so that $P_{KZ}^{\oplus N}/M$ is st. filtered.

Pf. $\mathrm{Ind}\,\mathrm{Res}\,M/M$ st. filtered $\Leftrightarrow$
$\mathrm{Ext}^1(\mathrm{Ind}\,\mathrm{Res}\,M/M, \nabla(\mu)) = 0 \ \forall \mu \Leftrightarrow$
$\mathrm{Hom}(\mathrm{Ind}\,\mathrm{Res}\,M, \nabla(\mu)) \to \mathrm{Hom}(M, \nabla(\mu))$ is surjective $\Leftrightarrow \mathrm{Hom}(M, \mathrm{Ind}\,\mathrm{Res}\,\nabla(\mu)) \to \mathrm{Hom}(M, \nabla(\mu))$ is surjective, but this is true, as $\mathrm{Ind}\,\mathrm{Res}\,\nabla(\mu) \to \nabla(\mu)$ is surjective which is seen by (by taking duals).

Lemma 8.11. If T is torsion in $\mathcal{O}_c$ and P is projective, then $\mathrm{Ext}^1(T, P) = 0$.

Proof. $P \hookrightarrow P_{KZ}^{\oplus N}$, $P_{KZ}^{\oplus N}/P$ standardly filtered by L. 8.9, 8.10. So by L. 8.9, ~~we have~~ $P_{KZ}^{\oplus N}/P \hookrightarrow P_{KZ}^{\oplus N'}$. Thus by taking duals, we have $0 \to {}^{\dagger}P_{KZ}^{\oplus N'}/K \to {}^{\dagger}P_{KZ}^{\oplus N} \to P^{\dagger} \to 0$.

So

$$\underbrace{\mathrm{Hom}\left(P^{+\oplus N'}_{K2}/K, T^+\right)}_{0} \to \mathrm{Ext}^1(P^+, T^+) \to \underbrace{\mathrm{Ext}^1(P^{+\oplus N}_{K2}, T^+)}_{0}$$

and thus $\mathrm{Ext}^1(P^+, T^+) = 0$, so $\mathrm{Ext}^1(T, P) = 0$.

It remains to deduce that

$KZ : \mathcal{O}_c/\mathcal{O}_c^{tor} \to \mathrm{Rep}\,\mathcal{H}_q(W)$ is an

equivalence. For this it suffices to show

that $\overline{P}_{KZ}$ is a progenerator of $\mathcal{O}_c/\mathcal{O}_c^{tor}$,

which is clear. ∎

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<u>Thm 8.12.</u> KZ is fully faithful

on projectives.

<u>Pf.</u> let $P_1, P_2 \in \mathcal{O}_c$ be two projectives.

~~Then~~ let $\overline{P_1}, \overline{P_2}$ be their images in $\mathcal{O}_c/\mathcal{O}_c^{tor} \cong \mathrm{Rep}\,\mathcal{H}_q$.

Then $$\mathrm{Hom}(\overline{P_1}, \overline{P_2}) = \varinjlim_{\substack{X \subset P_1 : P_1/X \in \mathcal{O}_c^{tor} \\ Y \subset P_2 \quad Y}} \mathrm{Hom}(X, P_2/Y)$$

But P_2 is free over $\mathbb{C}[\mathfrak{h}]$, so ~~X~~ $Y = 0$.

Thus $$\mathrm{Hom}(\overline{P_1}, \overline{P_2}) = \varinjlim_{\substack{X \subset P_1 \\ P_1/X \in \mathcal{O}_c^{tor}}} \mathrm{Hom}(X, P_2).$$

Now, ~~$0 \to \mathrm{Hom}(X, P_2)$~~ we have an exact sequence

$$\underset{0}{\underset{\parallel}{\mathrm{Hom}(P_1/X, P_2)}} \to \mathrm{Hom}(P_1, P_2) \to \mathrm{Hom}(X, P_2) \to \underset{0}{\underset{\parallel}{\mathrm{Ext}^1(P_1/X, P_2)}} \to 0.$$

By Lemma 8.11

so $\mathrm{Hom}(P_1, P_2) \cong \mathrm{Hom}(X, P_2)$

∎

<u>Example.</u> $W = S_n$, $c = \frac{1}{n}$, $q = e^{\frac{2\pi i}{n}}$

Then it's known that $\mathcal{H}_q^{(S_n)}$-mod looks like this: the Specht modules S_λ (defined similarly to S_n-case) are simple unless λ is a hook, and there is a separate block consisting of the hooks, other that $\lambda = (n)$. (So # of simples for $\mathcal{H}_q(S_n)$ is $p(n) - 1$, one less than for S_n).

This implies that $\mathcal{O}_{tor} \cong Vec$, spanned by $L(\mathbb{C}) = \mathbb{C}$. Same for $c = \frac{r}{n}$, $(r, n) = 1$, $r > 0$.

One can show that ~~the nontri~~ $\mathcal{O}_c$ also has $L(\lambda) = \Delta(\lambda)$ unless λ is a hook, and one nontrivial block corr. to hooks. In this block, we have

$\lambda_i = (n - i, \underbrace{1, \dots, 1}_{i})$, and in K_0,

$\Delta(\lambda_i) = L(\lambda_i) + L(\lambda_{i+1})$, $i < n$, $\Delta(\lambda_n) = L(\lambda_n)$.

This is the simplest nontriv. example of highest weight category, $\mathcal{C}(n)$, which we discussed before.