Lecture 9.

Quantum Schur-Weyl duality.

Let $n\geq 1$. Consider the operator $R:\mathbb{C}^m\otimes\mathbb{C}^m \circlearrowleft$ given by the formula

$$R(e_i\otimes e_j)=\begin{cases} q\,e_i\otimes e_j, & i=j\\ e_i\otimes e_j+(q-q^{-1})e_j\otimes e_i, & i<j\\ e_i\otimes e_j, & i>j\end{cases}\qquad (q\in\mathbb{C}^*)$$

e.g. for $n=2$
$$R=\begin{pmatrix} q & & & \\ & 1 & q-q^{-1} & \\ & & 1 & \\ & & & q\end{pmatrix}$$

(Note that $R|_{q=1}=1$).

Then $\check{R}=\sigma R:\mathbb{C}^m\otimes\mathbb{C}^m\circlearrowleft$ (where σ is the permutation) satisfies $(\check{R}-q)(\check{R}+q^{-1})=0$.

E.g. for $n=2$
$$\check{R}=\begin{pmatrix} q & & & \\ & q-q^{-1} & 1 & \\ & 1 & 0 & \\ & & & q\end{pmatrix},$$
so it has eigenvalues q (3 times) and $-q^{-1}$.

Also, R satisfies the QYBE:
$$R^{12}R^{13}R^{23}=R^{23}R^{13}R^{12},$$
so $\check{R}$ satisfies
$$\check{R}^{12}\check{R}^{23}\check{R}^{12}=\check{R}^{23}\check{R}^{12}\check{R}^{23}.$$

This implies, that the assignment

$T_i\mapsto \check{R}^{i\,i+1}$ defines a representation of the Hecke algebra $H_n(q)$ on $V^{\otimes n}$, where $V=\mathbb{C}^m$. This is a deformation of the permutation rep of S_n.

For $q=1$, the centralizer of $H_n(q)=\mathbb{C}S_n$ in $V^{\otimes n}$ is the image of $\mathcal{U}(\mathfrak{gl}(V))$, by the classical Schur-Weyl duality. Let us discuss the q-deformation of this statement. For this purpose we need to introduce the quantum alg. group $GL_q(m)$.

Recall that the algebraic group $GL(m)$ is studied via its algebra of functions $\mathcal{O}(GL(m))$, which is a commutative Hopf algebra. So let's define a Hopf algebra $\mathcal{O}_q(GL(m))$, which is no longer commutative. Recall that $\mathcal{O}(GL(n))$ is a localization of $\mathcal{O}(\mathrm{Mat}_m)$, and $\mathcal{O}(\mathrm{Mat}_n)$ is generated by the matrix elements of $L=(l_{ij})$, with commutation relation $[l_{ij}, l_{i'j'}]=0$, which can be written as $L^{13}L^{23}=L^{23}L^{13}$, where $L=\sum E_{ij}\otimes l_{ij}$. The coproduct is defined by $\Delta(l_{ij})=\sum l_{ik}\otimes l_{kj}$

($l_{ij}(gh)=\sum_k l_{ik}(g)\, l_{kj}(h)$, matrix product) so $\Delta(L) = L^{12}L^{13}$.

Now define the bialgebra $\mathcal{O}_q(\mathrm{Mat}_m)$ by generators $L=(l_{ij})$ as before, but with the commutation relation

$$R^{12}L^{13}L^{23} = L^{23}L^{13}R^{12},$$

and let the coproduct be as before. The localization $\mathcal{O}_q(GL(m))$ is defined in the same way as $\mathcal{O}(GL(m))$, by adjoining the inverse L^{-1} such that $LL^{-1} = L^{-1}L = 1$.

This localization is a Hopf algebra, with the antipode given by $S(L) = L^{-1}$!

Now, recall that algebraic representations of $GL(m)$ are the same things as $\mathcal{O}(GL(m))$-comodules. Such a comodule is a space Y together with an map $\rho_Y : Y \longrightarrow Y \otimes \mathcal{O}(GL(m))$ i.e. $\rho_Y(y) = L_Y(y \otimes 1)$, $L_Y \in \mathrm{End}(Y) \otimes \mathcal{O}(GL(m))$.

E.g. we have the vector representation $V = V$ defined by $L_V = L$. In the q-case, define the "vector representation" in the same way. This gives rise to a comodule$/\mathcal{O}_q$ due to the coproduct rule for L.

The category of $\mathcal{O}_q$-comodules is a tensor category, since $\mathcal{O}_q$ is a Hopf algebra. In particular, $V^{\otimes n}$ is a comodule, with

$$L_{V^{\otimes n}} = L^{1,n+1} \cdots L^{n,n+1}$$

Moreover, one can show that $\mathcal{O}_q$-comod is a braided tensor category, i.e. we have a functorial isomorphism $\beta_{W,U} : W \otimes U \to U \otimes W$, satisfying the hexagon axiom. Then $W^{\otimes n}$ carries an action of the ~~braid~~ group B_n. We will not construct the braiding (as we won't need it) but will say what it is if $W, U = V$. Then $\beta_{W,U} = R^\vee$.

<u>Exercise</u>. Check that $R^\vee : V \otimes V \circlearrowleft$ is a comodule morphism deforming the

permutation morphism.
So we get

<u>Proposition 9.1</u> The Hecke algebra $H_n(q)$ acts on $V^{\otimes n}$ by automorphisms of the O_q-comodule structure. Moreover, this action comes from the action of the braid group B_n on $V^{\otimes n}$.

This allows us to describe the centralizer of $H_n(q)$ in $V^{\otimes n}$ for generic q. Namely, let O_q^* be the algebra dual to the coalgebra O_q. Then V and $V^{\otimes n}$ may be regarded as O_q^*-modules. So we have the algebra $S_{q,m}(n)$, the image of O_q^* in $End(V^{\otimes n})$, which is called the q-Schur algebra. And we have

<u>Theorem 9.2.</u> (1) $S_{q,m}(n)$ commutes with $H_n(q)$ in $End(V^{\otimes n})$.

(2) If q is not a root of unity then $S_{q,m}(n)$ and $H_n(q)$ are semisimple and

are centralizers of each other. So we have a decomposition

$$V^{\otimes n} = \bigoplus_{\lambda : |\lambda| = n} W_q(\lambda) \otimes S_q(\lambda),$$

where $S_q(\lambda)$ are irreducible $H_n(q)$-modules, and $W_q(\lambda)$ are irreducible $\mathcal{O}_q$-comodules (or $\mathcal{O}_q^*$-modules, or $S_{q,m}(n)$-modules).

<u>Proof.</u> (1) is clear from the above.

(2) We will give a proof for generic q. (or over $\mathbb{C}(q)$). It is known that $H_n(q)$ is a flat deformation of $\mathbb{C}S_n$. So for generic q it is semisimple, with irreducible modules $S_q(\lambda)$ deforming the irreducible (Specht) modules $S(\lambda)$ for S_n. We have a homomorphism $\mathcal{O}_q^* \longrightarrow \mathrm{End}_{H_n(q)}(V^{\otimes n})$, which is flat in q and surjective for $q=1$. Hence it's surjective for generic q, and we are done. ∎

To make an analogy with the $q=1$ case described in Losev's lecture, note that the algebra $\mathcal{O}_q^*$ has a dense subalgebra U, which is the q-universal enveloping algebra of $gl(V)$ with (over $\mathbb{C}[q,q^{-1}]$) divided powers. This algebra is defined in the same way as usual enveloping algebra with divided powers, except all the factorials must be replaced by q-factorials: $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$, $[n]_q! = [n]_q [n-1]_q \cdots [1]_q$, and all binomial coefficients with q-binomial coefficients. This algebra is called the Lusztig's form of the quantum enveloping algebra.

<u>Corollary 9.3.</u> The assignment $(V = \mathbb{C}^m)$ $M \mapsto \mathrm{Hom}_{\mathcal{O}_q^*}(V^{\otimes n}, M) = Sh_m(M)$ is a functor from $\mathcal{O}_q$-comod to $\mathcal{H}_q(n)$-mod. If q is not a root of 1 This functor is an equivalence between the category of $\mathcal{O}_q(GL(m))$-comodules which occur in $V^{\otimes n}$ and the category of $\mathcal{H}_q(n)$-modules. For q root of unity, this functor has similar properties to characteristic p.