Lecture 2. Nakajima quiver varieties

1) Quivers & rep-s.
2) Nakajima q.v-s.
3) Examples
4) Poisson brackets
5) Constr-n of integr. rep-s
6) Sympl-c resol-ns & LMN isom-s

vert-s, arrows - fin sets

2.1) Quiver Q = or. graph, $Q=(Q_0, Q_1, t, h: Q_1 \to Q_0)$

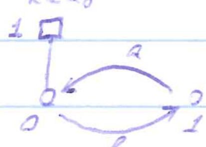
tail & head

Framed rep-s: $v, w \in \mathbb{Z}_{\geqslant 0}^{Q_0} \rightsquigarrow$ vect. spaces V_k, W_k, $\dim V_k = v_k$, $\dim W_k = w_k$, $k \in Q_0$

$R (= R(Q,v,w)) := \bigoplus_{a \in Q_1} \mathrm{Hom}_{\mathbb{C}}(V_{t(a)}, V_{h(a)}) \oplus \bigoplus_{k \in Q_0} \mathrm{Hom}(W_k, V_k)$

$\circlearrowleft$

$G (= GL(v)) = \prod_{k \in Q_0} GL(V_k)$

Example:  $(w=\epsilon_0) \rightsquigarrow R = \mathrm{Hom}(V_0,V_1) \oplus \mathrm{Hom}(V_1,V_0) \oplus V_0$

2.2) $G \curvearrowright R \rightsquigarrow G \curvearrowright T^*R = R \oplus R^*$ has canon sympl. form

$\rightsquigarrow$ moment map $\mu: T^*R \longrightarrow \mathfrak{g}^* \longleftrightarrow \mu^*: \mathfrak{g} \longrightarrow \mathbb{C}[T^*R]$

$$\begin{tikzcd} \mathfrak{g} \arrow[rr] \arrow[dr, "x \mapsto x_R"'] & & \mathbb{C}[T^*R] \\ & \mathrm{Vect}(R) \arrow[ur, hook] & \end{tikzcd}$$

G-equiv & $\{\mu^*(x), \cdot\} = x_{T^*R}$, $\{\cdot,\cdot\}$ - Poisson br-t on $\mathbb{C}[T^*R]$

quiv. var-ty $\mathcal{M}_\lambda^\theta(v,w) \leftsquigarrow \theta \in \mathbb{Z}^{Q_0}$, $\lambda \in \mathbb{C}^{Q_0}$

$\theta \rightsquigarrow \theta: G \to \mathbb{C}^\times$, $(g_k) \mapsto \prod_k \det(g_k)^{\theta_k}$

$\lambda \rightsquigarrow \lambda: \mathfrak{g} \to \mathbb{C}$ $(x_k) \mapsto \sum_k \lambda_k \mathrm{tr}(x_k)$

GIT-quot-t: $\mathcal{M}_\lambda^\theta(v,w) = \mu^{-1}(\lambda)^{\theta\text{-}ss} /\!\!/ G$

affine $\mathcal{M}_\lambda^0(v,w) \xleftarrow[\text{proj. morph}]{\pi} \underbrace{\mathcal{M}_\lambda^\theta(v,w)}_{\text{smooth if } G \curvearrowright \mu^{-1}(\lambda)^{\theta\text{-}ss} \text{ freely}}$

Say (λ,θ)-generic (descr. below)

Rem: • lin. alg. interpt. of μ

$$(x_a, x_{a^*}, i, j) \in \bigoplus_a \left[Hom(V_{t(a)}, V_{h(a)}) \oplus Hom(V_{h(a)}, V_{t(a)})\right] \oplus \bigoplus_k \left[Hom(W_k, V_k) \oplus Hom(V_k, W_k)\right]$$

$$\downarrow \mu$$

$$(\mu_k)_{k \in Q_0} \in \mathfrak{g}, \quad \mu_k = \sum_{a,\, h(a)=k} x_a x_{a^*} - \sum_{a,\, t(a)=k} x_{a^*} x_a + i_k j_k; \quad \mu = [x_a, x_{a^*}] + ij$$

• interpr. of $R^{\theta-ss}$ – all (x_a, x_{a^*}, i_k, j_k) s.t. (for θ gener.)

$\nexists$ proper subrep. $(V'_k) \subset (V_k)$ (stab. w.r.t. x_a, x_{a^*}) s.t.

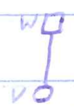
either $\theta \cdot v' \le \theta \cdot v$ & $j_k(V'_k) = 0\ \forall k$ (always false if $\theta_k < 0\ \forall k$)

or $\theta \cdot v' \ge \theta \cdot v$ & $V'_k \supset im\, i_k\ \forall k$ (always false if $\theta_k > 0\ \forall k$)

• independence of orient.

$Q \rightsquigarrow$ change orient. of $a \rightsquigarrow Q'$

$$R \xrightarrow[(x_a, x_{a^*}) \mapsto (x_{a^*}, -x_a)]{\sim} R' \rightsquigarrow \text{id-n of quiv. var. } \mathcal{M}^\theta_\lambda(v,w).$$

2.3) Ex 1: 

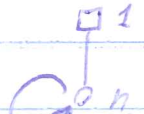
$w \ge v$, $\theta > 0$

$R = Hom(W,V) \oplus Hom(V,W)$ $\quad \mu(i,j) = ij$

$(i,j) \in R^{\theta-ss} \iff j$ is inj

$\mu^{-1}(0)^{\theta-ss} = \{(i,j) \mid j \text{ is inj},\ i: W/im\, j \to V\}$

$\mu^{-1}(0)^{\theta-ss} = T^*Gr(v,w)$

Ex 2: 

$R = End(V)^{\oplus 2} \oplus V \oplus V^*$, $\theta < 0$

$\mu(X,Y,i,j) = [X,Y] + ij$

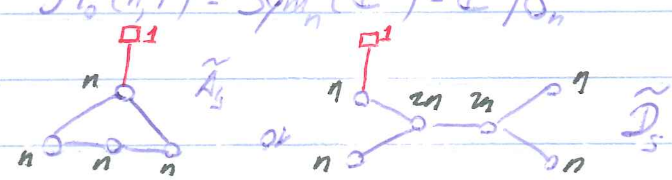
$R^{\theta-ss} = \{(X,Y,i,j) \mid \mathbb{C}\langle X,Y\rangle i = V\}$

$(X,Y,i,j) \in \mu^{-1}(0)^{\theta-ss} \Rightarrow j = 0$, so $\mathcal{M}^\theta_0(n,1) = Hilb_n(\mathbb{C}^2)$

$$\downarrow \pi \text{ - Hilbert-Chow morphism}$$

$$\mathcal{M}^0_0(n,1) = Sym_n(\mathbb{C}^2) = \mathbb{C}^{2n}/\mathfrak{S}_n$$

Ex 3 (gen-n of 2)

Q-ext-d Dynkin diagr., e.g. 

$v = n\delta$ (δ-indec. imag. root), $w = \epsilon_0$, generic θ

$Q \rightsquigarrow \Gamma_1 \subset SL_2(\mathbb{C}) \rightsquigarrow \Gamma_n \subset Sp(\mathbb{C}^{2n})$ See addendum 1

$\mathcal{M}_0^0(n\delta, \epsilon_0) = \mathbb{C}^{2n}/\Gamma_n$

$\uparrow$ - resol-n of sing. (some $\theta \rightsquigarrow Hilb_n(\widetilde{\mathbb{C}^2/\Gamma_1})$)

$\mathcal{M}_0^\theta(n\delta, \epsilon_0)$

2.4) A - Poisson alg., $G \curvearrowright A$ + $\mu^*: \mathfrak{g} \to A$ - comoment map

$\rightsquigarrow$ Hamilt. red-n $A///_\lambda G = [A/\underbrace{A\{\mu^*(x) - \langle\lambda, x\rangle, x\in\mathfrak{g}\}}_{I}]^G$

Poisson alg.

$\{a+I, b+I\} = \{a,b\} + I$

Example: $\mathbb{C}[\mathcal{M}_\lambda^0(v,w)] = \mathbb{C}[T^*R]///_\lambda G$

$\mathcal{M}_\lambda^\theta(v,w)$ - glued from H.R. compat. w. $\{\cdot,\cdot\}$'s $\rightsquigarrow \mathcal{M}_\lambda^\theta(v,w)$ - Poisson scheme

(λ,θ)-generic $\Rightarrow \mathcal{M}_\lambda^\theta(v,w)$ is symplect.

Rem (grading/$\mathbb{C}^\times$-action) $\mathbb{C}^\times \curvearrowright T^*R$ (scaling), μ is homog. $\rightsquigarrow G \curvearrowright \mu^{-1}(0) \rightsquigarrow$

$G \curvearrowright \mathcal{M}_0^\theta(v,w)$ - compat. w. $\{\cdot,\cdot\}$: $t.\{\cdot,\cdot\} = t^{-2}\{\cdot,\cdot\}$

equiv ($\theta=0$): $\mathbb{C}[\mathcal{M}_0^0(v,w)]$ is graded w. $\{\cdot,\cdot\}$ decr. tot. deg. by 2

Assume Q w/o loops

2.5) $Q \rightsquigarrow \mathfrak{g}(Q)$ - (symm-c) Kac-Moody Lie alg. w. Dynkin diagram = unorient. Q

$\circ \rightleftarrows \circ \rightsquigarrow \hat{\mathfrak{sl}}_2$; $\mathfrak{h} \subset \mathfrak{g}(Q)$ - Cartan, $\alpha^k, \omega^k \in \mathfrak{h}^*$, $k \in Q_0$, - simple roots & fund. wts;

$w \rightsquigarrow \omega := \sum_{k\in Q_0} w_k \omega^k$, $v \mapsto \nu := \omega - \sum_{k\in Q_0} v_k \alpha^k$.

Goal: realize integr. irrep L_ω w. high. wt ω in terms of $\mathcal{M}_0^\theta(v,w)$ (variabl. v)

Rem: $\mathbb{C}^\times \curvearrowright \mathcal{M}_0^\theta(v,w)$ contr-g to $\pi_v^{-1}(0)$ - Lagrang. subvar.

$\Rightarrow H_*(\mathcal{M}_0^\theta(v,w)) = H_*(\pi_v^{-1}(0))$, top deg = $\dim_{\mathbb{C}} \mathcal{M}_0^\theta(v,w)$

$H_{top} = \mathbb{C}^{comp}$, comp = {irred. comp-s of $\pi_v^{-1}(0)$}

Thm (Nakajima) $\mathfrak{g}(Q) \curvearrowright \bigoplus_v H_{top}(\mathcal{M}_0^\theta(v,w)) \simeq L_\omega$

$H_{top}(\mathcal{M}_0^\theta(v,w)) = L_\omega[\lambda]$ -wt. space

Example: $\overset{\square w}{\underset{\circ v}{|}}$, $\theta > 0$. $\mathcal{M}_0^\theta(v,w) = T^*Gr(v,w)$, $\pi_v^{-1}(0) = Gr(v,w)$

Op-rs e, f: $\begin{array}{ccc} & Fl(v,v+1,w) & \\ {}^{\pi_1}\swarrow & & \searrow{}^{\pi_2} \\ Gr(v,w) & & Gr(v+1,w) \end{array}$ $\quad e|_{H_{top}(\pi_{v+1}^{-1}(0))} = \pi_{1*}\pi_2^*$

$f|_{H_{top}(\pi_v^{-1}(0))} = \pi_{2*}\pi_1^*$

Rem: descr-n of gen. (λ,θ): $\nexists\ v' \in \mathbb{Z}_{\geq 0}^{Q_0}$, $v'_k \leq v_k\ \forall k$, $\sum_k v'_k \alpha^k$-root for $\mathfrak{g}(Q)$ s.t. $v' \cdot \theta = v' \cdot \lambda = 0$. see addendum 2

$(\sum_k v'_k \theta_k)$

Cor: $H_*(\mathcal{M}_\lambda^\theta(v,w))$ are canon ident. for (λ,θ) (reason: using hyperkähler red-n form-m may use $\theta \in \mathbb{R}^{Q_0}$, compl. to gener. loc in $\mathbb{C}^{Q_0} \times \mathbb{R}^{Q_0}$ has real codim 3).

6) ν-domin $\Rightarrow \pi: \mathcal{M}_0^\theta(v,w) \longrightarrow \mathcal{M}_0^0(v,w)$ -resol-n of sing-s (Nakajima)

+Poisson iso $\Rightarrow$ sympl resol-n of sing ☺

Nice prop-s • $H^i(\mathcal{O}_{\mathcal{M}_0^\theta(v,w)}) = 0,\ i > 0$ $(\mathbb{C}[\mathcal{M}_0^\theta(v,w)] = \mathbb{C}[\mathcal{M}_0^0(v,w)])$

• π is semismall: $\operatorname{codim}\{x \in \mathcal{M}_0^0(v,w) \mid \dim \pi^{-1}(x) \geq n\} \geq 2n$

Q: what if ν not domin-t?

A: have isom-s constit. $\underbrace{W(Q)}_{\text{Weyl grp of } \mathfrak{g}(Q)}$-action

$\theta, \lambda \in \mathfrak{h}^*$, $\sigma \in W(Q) \rightsquigarrow \sigma\theta, \sigma\lambda$

$\sigma \cdot v$: $(s_k \cdot v)_\ell = \begin{cases} v_\ell, & \ell \neq k \\ w_k + \sum\limits_{i \underset{Q}{\rightleftarrows} k} v_i - v_k \end{cases}$

Claim: (Lusztig-Maffei, Nakajima) $\mathcal{M}_\lambda^\theta(v,w) = \mathcal{M}_{\sigma\lambda}^{\sigma\theta}(\sigma \cdot v, w)$

Example: $\overset{\square w}{\underset{\circ v}{|}}$, $\lambda = 0$, $\theta > 0$

$\mathcal{M}_0^\theta(v,w) = T^*Gr(v,w)$, $Gr(v,w) = \{v\text{-dim-l subsp. in } W\}$

$\mathcal{M}_0^{-\theta}(w-v,v) = T^*Gr(v,w)$ $\quad = \{w-v\text{-dim-l quot-t of } W\}$

$\lambda \neq 0$ - exercise

Spec. case: $\lambda_i = 0$, $\theta_i > 0$: $\underline{R}$ - ~~part of~~ summand of R, all maps not incl.

V_i: $\underline{G}$ - simil., $\underline{\lambda}, \underline{\theta}$. (assume k is sink)

$$\mathcal{M}_\lambda^\theta(v,w) = [\underbrace{T^*R /\!\!/\!\!/_0^{\theta_i} GL(V_k)}_{\text{Hamilt. red-n}}] /\!\!/\!\!/_{\underline{\lambda}}^{\underline{\theta}} \underline{G} = [T^*Gr(v,w) \times \underline{R}] /\!\!/\!\!/_{\underline{\lambda}}^{\underline{\theta}} \underline{G}$$

$\mathcal{M}_{\sigma\lambda}^{\sigma\theta}(v,w)$ = same (rem. $\underline{\lambda}$ liter. same, $\underline{\theta}$ changed due to diff. interpr. of $Gr(v,w)$), k becomes source

Gen. case: exercise.

Addendum 1: More on isomorphism $\mathcal{M}_0^0(n\delta, \epsilon_0) = \mathbb{C}^{2n}/\Gamma_n$

• $n=1$: can ignore framing: $(x_a, x_{a^*}, i, j) \in \mu^{-1}(0) \Rightarrow ij = 0 \Rightarrow \mathcal{M}_0^0(\delta, \epsilon_0) = \mathcal{M}_0^0(\delta, 0)$.

McKay corresp: $\Gamma_1 \subset SL_2(\mathbb{C}) \rightsquigarrow$ quiver $\overline{Q}$ w $\overline{Q}_0 = \{0, 1, .. r\}$, where

$\operatorname{Irr} \Gamma_1 = \{N_0, N_1, .. N_r\}$, $N_0 = \text{triv}$, $\#\{a : i \to j\} = \dim \operatorname{Hom}_{\Gamma_1}(\mathbb{C}^2 \otimes N_i, N_j)$

Then $\overline{Q}$ is $\underbrace{\text{double}}_{\text{add oppos. to } \forall \text{ arrow}}$ of affine quiver Q & $\delta = (\dim N_i)_{i=0}^r$.

So $T^*R = \operatorname{Rep}(\overline{Q}, \delta) = \operatorname{Hom}_{\Gamma_1}(\mathbb{C}^2 \otimes \mathbb{C}\Gamma_1, \mathbb{C}\Gamma_1) = \operatorname{Rep}(\mathbb{C}\langle x,y \rangle \# \Gamma_1)$

~~var-ty param. s/simple rep-s of~~ alg. homom $\mathbb{C}\langle x,y\rangle \# \Gamma_1 \to \operatorname{End}(\mathbb{C}\Gamma_1)$

whose restr-n to $\mathbb{C}\Gamma_1$ is left mult-n

~~$\operatorname{Rep}(\mathbb{C}\langle x,y \rangle \# \Gamma_1) /\!\!/ G = \{\text{s/simple rep-s / iso}\}$~~

$$\begin{array}{ccc} \operatorname{Rep}(\mathbb{C}[x,y] \# \Gamma_1) & \subset & \operatorname{Rep}(\mathbb{C}\langle x,y \rangle \# \Gamma_1) \\ \| & & \| \\ \mu^{-1}(0) & \subset & T^*R \end{array}$$

$$\mu^{-1}(0) /\!\!/ G = \operatorname{Rep}(\mathbb{C}[x,y] \# \Gamma_1) /\!\!/ G = \{\text{s/simple rep-s / iso}\} \xrightarrow[\sim]{\text{centr. character}} \mathbb{C}^2/\Gamma_1.$$

• $n > 1$: $\mu_1^{-1}(0)^n \hookrightarrow \mu^{-1}(0)$: $(r_1, \dots r_n) \mapsto r_1 \oplus .. \oplus r_n$

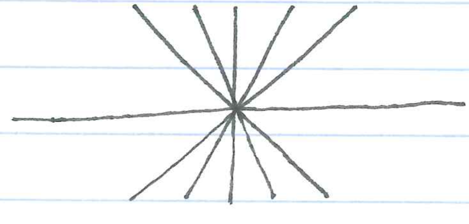
($\mu_1^{-1}(0)$ for $v = \delta$, $w = 0$)

$$\mathbb{C}^{2n}/\Gamma_n = \mu_1^{-1}(0)^n / \mathfrak{S}_n \xrightarrow{\sim} \mu^{-1}(0) /\!\!/ GL(n\delta)$$

Addendum 2:

Picture for generic θ

(i.e. $(0,\theta)$-gener.) $\Gamma_1 = \mathbb{Z}/2\mathbb{Z}$, $n = 3 \rightsquigarrow Q =$ 



Lecture 4. Quantization of quiver varieties

1) Quant-n of grad alg-s & quant. Hamilt. red-n (qHR)

2) Isomorphism of qHR & spher. SRA.

3) Microlocal quant-n & qHR

5 ~~4~~) Supports

6 ~~5~~) Local-n thms

7 ~~6~~) Transl-n fun-rs

4 ~~7~~) Quant-n of LMN isom-s.

4.1) $A=\bigoplus_{i\in\mathbb{Z}} A^i$ - grad. Poiss. alg. w. $\{A^i,A^j\}\subset A^{i+j-d}$ $\forall i,j$ $(d\in\mathbb{Z}_{>0})$

Quant-n: $\mathcal{A}=\bigcup_{i\in\mathbb{Z}}\mathcal{A}^{\leq i}$ - filt. assoc. alg. w. $[\mathcal{A}^{\leq i},\mathcal{A}^{\leq j}]\subset\mathcal{A}^{\leq i+j-d}$ & $\operatorname{gr}\mathcal{A}\xrightarrow{\sim}A$ (isom of grad. Poiss. alg.) + filtr-n is $\underbrace{\text{compl. \& separ-d}}$ (w.r.t. topol. w. basis $\mathcal{A}^{\leq i}$)

vacuous if $\mathcal{A}^{\leq -1}=0$

Ex: • $eH_{1,c}e$ - quant-n of $S(V)^\Gamma$ $(d=2)$

• $D(R)$ - quant-n of $\mathbb{C}[T^*R]$

• filtr-n by order of d.o. $(\deg R^*=0, \deg R=1)\rightsquigarrow d=1$

• Bernstein fn $(\deg R^*=\deg R=1)\rightsquigarrow d=2$

Q: quantize $A=\mathbb{C}[T^*R]/\!/\!/_0 G=[\mathbb{C}[T^*R]/\underbrace{\mathbb{C}[T^*R]\mu^*(\mathfrak{g})}_{I}]^G$

recipe: by quant-ng red-n $\mathbb{C}[T^*R]\rightsquigarrow D(R)$

$\mu^*:\mathfrak{g}\to\mathbb{C}[T^*R]\rightsquigarrow\Phi:\mathfrak{g}\to D(R)$, $\Phi(x):=x_R\in Vect(R)\subset D(R)$

quant. comom. map: G-equiv + $[\Phi(x),\bullet]=x_{D(R)}$

$\lambda\in\mathbb{C}^{Q_0}\rightsquigarrow\tilde{\mathcal{A}}_\lambda=[D(R)/\underbrace{D(R)\{\Phi(x)-\langle\lambda,x\rangle, x\in\mathfrak{g}\}}_{\mathcal{I}}]^G$

Product: $(a+\mathcal{I})(b+\mathcal{I})=ab+\mathcal{I}$.

Rem: $\operatorname{gr}\mathcal{I}\supset I$

$=$ $\Longleftarrow$ $\operatorname{codim}\mu^{-1}(0)=\dim\mathfrak{g}$

quiver case: $\Updownarrow$ combin. cond-n on v (Crawley-Boevey)

$\Uparrow$ $\leftarrow$ Q finite/affine

$\rightarrow$ domin.

$gr\,\mathcal{I} = I \Rightarrow gr\,\mathcal{D}(R)/\mathcal{I} = \mathbb{C}[T^*R]/I \Rightarrow gr\,\tilde{\mathcal{A}}_\lambda = A.$

Rem: φ, φ' - quant. comom. maps $\Rightarrow$ [center $\mathcal{D}(R) = \mathbb{C}$] $\varphi(x) - \varphi'(x) =$
$= \langle \chi, x \rangle$, $\chi \in \mathbb{C}^{Q_0}$.

e.g $\varphi(x) = x_R$, $\varphi'(x) = x_{R^*}$, $\varphi^{sym}(x) = \frac{1}{2}(x_R + x_{R^*})$.

4.2) Q = ext-d Dynkin quiver, $v = n\delta$, $w = \epsilon_0 \rightsquigarrow \mathcal{M}_0^0(n\delta, \epsilon_\infty) = \mathbb{C}^{2n}/\Gamma_n$
$\Gamma_1 \subset SL_2(\mathbb{C}) \rightsquigarrow \Gamma_n = \mathfrak{S}_n \ltimes \Gamma_1^n \subset Sp(\mathbb{C}^{2n})$

q-ns of $\mathbb{C}[\mathbb{C}^{2n}/\Gamma_n]$: $\tilde{\mathcal{A}}_\lambda^{sym}$ = [slightly redef.] = $[\mathcal{D}(R)/\mathcal{D}(R)\{\varphi^{sym}(x) - \langle \lambda, x \rangle, x \in \mathfrak{g}\}]^G$
$eH_{1,c}e \quad (c = (k, c_1, \dots c_r))$

$Q_0 = \{0, 1, .. r\} \longleftrightarrow \Gamma_1$-irreps, $k \longmapsto N_k$, N_0 := triv.

$c \rightsquigarrow \lambda$: $\quad C := 1 + \sum_{i=1}^r c_i \sum_{\gamma \in S_i^0} \gamma$

$\lambda_k := \frac{1}{|\Gamma_1|} tr_{N_i} C$, $i = 1, \dots r$, $\qquad \lambda_0 := \frac{1}{|\Gamma_1|} tr_{N_0} C - \frac{1}{2}(1 + k)$.

Thm: (Holland, EG, GG, O, Gordon, EGGO; I.L.) - to be proved in Lec 7.
$\tilde{\mathcal{A}}_\lambda^{sym} = eH_{1,c}e$.

4.3) Q: quant-n of $\mathcal{M}_0^\theta(v, w)$?

• X - non-aff. Poisson var-ty w. $\mathbb{C}^\times \curvearrowright X$, $U \subset X$ - $\mathbb{C}^\times$-stab $\rightsquigarrow \mathcal{O}(U)$-graded

Def: (microlocal) quant-n $\mathcal{U}$ of X (i.e. of $\mathcal{O}$) - sheaf (in conic. topol: open = $\mathbb{C}^\times$-stab. open) of (compl & separ.) filt. Poiss. alg-s w. $gr\,\mathcal{U} = \mathcal{O}$.

Example: X-aff. w $\mathbb{C}[X]^i = 0$, $i < 0$.

quant-n of $\mathcal{O} \Leftrightarrow$ quant-n of X
$\mathcal{U} \longmapsto \Gamma(\mathcal{U})$
microloc-n $\longleftarrow \mathcal{A}$
enough $\mathcal{U}(X_f)$, f-homog., $X_f = \{x \in X \mid f(x) \neq 0\}$
suitable loc-n of $\mathcal{A}$.

$\mathcal{A} \rightsquigarrow$ Rees alg. $\mathcal{A}_\hbar := \bigoplus_{i \geq 0} \mathcal{A}^{\leq i} \hbar^i \subset \mathcal{A}[\hbar]$ - graded subalg.

$\mathcal{A}_\hbar/(\hbar) = gr\,\mathcal{A}$, $\quad \mathcal{A}_\hbar/(\hbar - z) = \mathcal{A}$, $z \neq 0$

$f \in \mathrm{gr}\,\mathcal{A} \rightsquigarrow$ homog. lift $\hat{f} \in \mathcal{A}_\hbar$, $\mathrm{ad}(\hat{f})^n \mathcal{A}_\hbar \subset \hbar^n \mathcal{A}_\hbar \Rightarrow \hat{f}$ is loc-le in
$\mathcal{A}_\hbar/(\hbar^n)\ \forall n \rightsquigarrow \mathcal{A}_\hbar/(\hbar^n)[\hat{f}^{-1}]$ - depends only on f.
$\mathcal{A}_\hbar[f^{-1}] := \varprojlim_n \mathcal{A}_\hbar/(\hbar^n)[\hat{f}^{-1}] \curvearrowleft \mathbb{C}^\times$

$\mathcal{A}[f^{-1}] := \{\underbrace{\mathbb{C}^\times\text{-fin}}_{\text{sums of semiinvar-s}} \text{ el-ts of } \mathcal{A}_\hbar[f^{-1}]\}/(\hbar-1)$ - compl. filt. alg. w $\mathrm{gr}\,\mathcal{A}[f^{-1}] = \mathbb{C}[X_f]$

$\downarrow$ sheafify

$\mathfrak{A}$. $\square$

Quant-n of $\mathcal{M}_0^\theta(v,w)$: $D(R) \rightsquigarrow$ sheaf $\mathcal{D} \rightsquigarrow \mathcal{D}|_{(T^*R)^{\theta\text{-ss}}} \rightsquigarrow$
$\mathcal{D}/\mathcal{D}\{\varphi(x) - \langle\lambda, x\rangle\}|_{(T^*R)^{\theta\text{-ss}}}$ - sheaf on $\mu^{-1}(0)^{\theta\text{-ss}} \xrightarrow[\rho]{} \mathcal{M}_0^\theta(v,w)$

$\mathcal{A}_\lambda^\theta := \rho_*\left[\mathcal{D}/\mathcal{D}\{\dots\}|_{(T^*R)^{\theta\text{-ss}}}\right]^G$

$G \curvearrowright \mu^{-1}(0)^{\theta\text{-ss}}$ - free $\Rightarrow \mathrm{gr}\,\mathcal{A}_\lambda^\theta = \mathcal{O} \Rightarrow [H^i(\mathcal{O}) = 0, i > 0]$
$\mathrm{gr}\,\Gamma(\mathcal{A}_\lambda^\theta) = \mathbb{C}[\mathcal{M}_0^\theta(v,w)]$, $H^i(\mathcal{A}_\lambda^\theta) = 0$, $i > 0$
Rel-n to $\tilde{\mathcal{A}}_\lambda$: $\tilde{\mathcal{A}}_\lambda \longrightarrow \Gamma(\mathcal{A}_\lambda^\theta)$
λ - domin $\Rightarrow$ $\xrightarrow{\sim}$
Gen-l λ: $\mathcal{A}_\lambda := \Gamma(\mathcal{A}_\lambda^\theta)$ - indep. of θ (see below)

4.4) Reminder: $\mathcal{M}_{\sigma\lambda}^{\sigma\theta}(\sigma\cdot v) \xleftarrow[\sigma]{\sim} \mathcal{M}_\lambda^\theta(v)$ $\quad \sigma \in W(Q)$
lifts to
$$\mathcal{A}_{\sigma\lambda}^{sym,\sigma\theta}(\sigma\cdot v) \xleftarrow{\sim} \mathcal{A}_\lambda^{sym,\theta}(v)$$
Either: str-re of quan-ns of sympl. resol-ns (R. B-Kaledin, I.L.) or
constr-n simil. to class. LMN iso-s:
1) $\begin{array}{l}\square\, w\\ |\\ \circ\, v\end{array}$
2) gen-l case via "red-n in stages"

4.5) Supports:

$\mathcal{A}$-quant-n of $A = \mathbb{C}[X]$; want reduce rep-ns of $\mathcal{A}$ to A-modules
affine

$\operatorname{gr}\mathcal{A} = A$

$\mathcal{M} \in \mathcal{A}\text{-}\underline{\text{mod}}$ cat-y of fin. gen mod-s

Def: good filtr-n on $\mathcal{M}$: $\mathcal{M} = \bigcup_{i\in\mathbb{Z}} \mathcal{M}^{\leq i}$ (compl. & separ.), compat. w. filtr-n on $\mathcal{A}$ & $\operatorname{gr}\mathcal{M}$ is fin. gen. A-module

depends on choice of filtr. but

$$\left.\begin{aligned} &\operatorname{Supp}(\mathcal{M}) := \operatorname{Supp}(\operatorname{gr}\mathcal{M}) \\ &CC(\mathcal{M}) = \sum_{\substack{Z\text{-irred. comp of} \\ \operatorname{Supp}(\mathcal{M})}} [\text{mult of } Z \text{ in } \operatorname{gr}\mathcal{M}]\cdot Z \end{aligned}\right\} \text{well-def.}$$

Non-affine setting:

$\mathcal{U}$-quant-n of X.

$\mathcal{U}$-Mod - cat-y of all $\mathcal{U}$-modules

$\cup$

$\mathcal{U}$-mod - cat-y of all mod-s w. good filtr-n ($\operatorname{gr}\mathcal{M} \in \operatorname{Coh}(X)$)

Supp & CC make sense

4.6) $\pi: \mathcal{M}_0^\theta(v) \longrightarrow \mathcal{M}_0'(v) := \operatorname{Spec}(\mathbb{C}[\mathcal{M}_0^\theta(v)])$ - resol-n of sing.

$\rightsquigarrow \pi_*: \operatorname{Coh}\mathcal{M}_0^\theta(v) \rightleftarrows \operatorname{Coh}\mathcal{M}_0'(v) : \pi^*$ adj-t pair

+ derived functors.

Q: analogs for $\mathcal{A}_\lambda^\theta(v), \mathcal{A}_\lambda(v)$

global sections $\Gamma (= \Gamma_\lambda^\theta)$ $\mathcal{A}_\lambda^\theta(v)\text{-mod} \longrightarrow \mathcal{A}_\lambda(v)\text{-mod}$

$Loc := \mathcal{A}_\lambda^\theta(v) \otimes_{\mathcal{A}_\lambda(v)} \bullet : \mathcal{A}_\lambda(v)\text{-mod} \longrightarrow \mathcal{A}_\lambda^\theta(v)\text{-mod}$

+ $R\Gamma: D^b(\mathcal{A}_\lambda^\theta(v)\text{-mod}) \longrightarrow D^b(\mathcal{A}_\lambda(v)\text{-mod}) \longleftarrow$ e.g. via Čech complex

complexes w. homol. in $\mathcal{A}_\lambda^\theta(v)$-mod

$LLoc: D^-(\mathcal{A}_\lambda(v)\text{-mod}) \longrightarrow D^-(\mathcal{A}_\lambda^\theta(v)\text{-mod})$

- often q-inverse equiv.-s

Thm (derived local-n; McGerty-Nevins) $R\Gamma$ is equi $\Leftrightarrow$ homol. dim $\mathcal{A}_\lambda(v) < \infty$

$\Rightarrow LLoc$ is q. inverse

4.4

conj: $\{\lambda \mid \text{hom.dim}\, \mathcal{A}_\lambda(v) < \infty\}$ = fin $\cup$ of aff. hyperplanes

≈ known = – · – · – · – · – subspaces

Q: When Γ & Loc equiv-s? – more subtle

$\{\theta \mid (0,\theta)\text{-generic}\}$ = compl. to hyperplanes in $\mathbb{R}^{Q_0}$

chamber: = conn. comp

$\theta \in C$ – chamber.

Thm (Braden-Proudfoot-Webster): $\forall \lambda\ \exists \lambda'$ s.t. $\lambda' - \lambda \in \mathbb{Z}^{Q_0}$ &

$\Gamma^\theta_{\lambda'+\chi}$ is equiv. $\forall \chi \in \mathbb{Z}^{Q_0} \cap C$. (see picture below)

~~I~~ Informal: Γ^θ_λ is equiv. $\forall \lambda$ "large enough"

Reason to want loc-n theory: geometry of $\mathcal{M}^\theta(v)$ is nicer than of $\mathcal{M}'(v)$

4.7 Appl-n – translation functors

~~$\lambda \in \mathbb{C}$~~ $\chi \in \mathbb{Z}^{Q_0} \rightsquigarrow$ line bunde $\mathcal{O}(\chi) = [\mathcal{O}_{\mu^{-1}(0)^{ss}}]^{G,\chi}$

Quant-n: $\mathcal{A}^\theta_{\lambda,\chi}(v) := [\mathcal{D}/\mathcal{D}\{\Phi(x) - \langle \lambda, x \rangle\}]^{G,\chi}$ – sheaf of

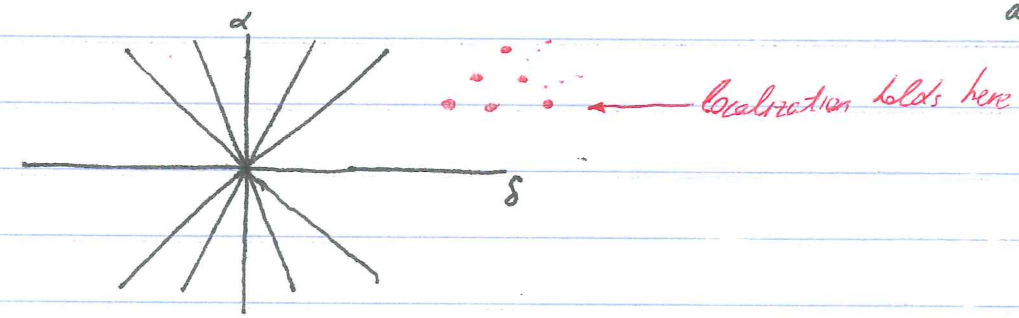
$\mathcal{A}^\theta_{\lambda+\chi}(v)$-$\mathcal{A}^\theta_\lambda(v)$-bimodules w. gr $\mathcal{A}^\theta_{\lambda,\chi}(v) = \mathcal{O}(\chi)$

$\rightsquigarrow$ equiv.: $\mathcal{A}^\theta_\lambda(v)\text{-mod} \xrightarrow{\sim} \mathcal{A}^\theta_{\lambda+\chi}(v)\text{-mod}$

So $\mathcal{A}^\theta_\lambda(v)$-mod depends only on θ & $\lambda + \mathbb{Z}^{Q_0}$

So $\mathcal{A}_\lambda(v), \mathcal{A}_{\lambda+\chi}(v)$ fin. hom. dim $\Rightarrow D^b(\mathcal{A}_\lambda(v)\text{-mod}) \xrightarrow{\sim} D^b(\mathcal{A}_{\lambda+\chi}(v)\text{-mod})$

Rem: Inform. Loc-n thm $\leftsquigarrow H^i(\mathcal{O}(\chi)) = 0$, $i > 0$, for χ "$\underset{\text{ample}}{\underline{\text{large}}}$, enough"



4.5

Lecture 7.
Procesi bundles & their deformations

1) Procesi bundles
2) Sketch of constr-n
3) Deform-s of P.b. & iso. btw SRA & qHR
4) Deformed derived McKay corresp
5) Appl-ns to P.b-s: classif-n & wreath Macdonald positiv-y.

1) $X_0 := \mathbb{C}^{2n}/\Gamma_n \xrightarrow{\sim} \mathcal{M}_0^0(n\delta, \epsilon_0)$ – $\mathbb{C}^\times$-equiv. isom (actions via dilations)

X – $\mathbb{C}^\times$-equiv. sympl-c resol-n, e.g. $X = \mathcal{M}_0^\theta(n\delta, \epsilon_0)$, $\pi: X \to X_0$.

Def: Procesi bundle on X = $\mathbb{C}^\times$-equiv. bundle $\mathcal{P}$ + ($\mathbb{C}^\times = \mathbb{G}_m$ – 1-dim torus)

$$\begin{array}{ccc} End(\mathcal{P}) & \xrightarrow[\mathbb{C}^\times\text{-equiv}]{\sim} & \mathbb{C}[\mathbb{C}^{2n}] \# \Gamma_n \\ \cup & & \cup \\ \mathbb{C}[X] & \xleftarrow[\pi^*]{\sim} & \mathbb{C}[\mathbb{C}^{2n}]^{\Gamma_n} \text{ -center} \end{array} \qquad (*)$$

& $Ext^i(\mathcal{P}, \mathcal{P}) = 0 \ \forall i > 0$.

Norm-n ($\mathcal{P}$ – P.b $\Rightarrow \mathcal{P} \otimes \mathcal{L}$ – P.b, $\mathcal{L} \in Pic(X)$)

$(*) \Rightarrow \mathcal{P}_x \underset{\Gamma_n}{\simeq} \mathbb{C}\Gamma_n \ \forall x \in X \Rightarrow \mathcal{P}^{\Gamma_n} =$ line bundle

Assume: $\mathcal{P}^{\Gamma_n} = \mathcal{O}_X$.

Example ($n=1$): have taut. bundle $\mathcal{T}$ on $X := \mathcal{M}_0^\theta(n\delta, \epsilon_0)$

$G = GL(n\delta)$-module $V = \bigoplus_{i \in Q_0} \mathbb{C}^{\delta_i} \otimes N_i^*$ – w. left action

$\rightsquigarrow$ G-equiv. bundle $\mathcal{V} = V \times X \to X \rightsquigarrow$ induced bundle $\mathcal{T}$ on $\mu^{-1}(0)^{\theta-ss}/G$

diag. action

$\mathcal{T}$ is P.b.

$\Gamma_1 = \{1\}$ – const-d by Haiman

Gen-l case: R. B & Kaledin

Appl-n: (derived McKay equiv.) $RHom_{\mathcal{O}_X}(\mathcal{P}, \bullet): D^b(Coh\, X) \to D^b(\mathbb{C}[\mathbb{C}^{2n}] \# \Gamma_n\text{-mod})$
$= D^b(Coh^{\Gamma_n}(\mathbb{C}^{2n}))$

2) Idea: reduction to char $p >> 0$

~~7.2) Sketch of constr-n~~

X-def / R = loc-n of $\mathbb{Z}$ $\rightsquigarrow$ $X_R / \operatorname{Spec} R$ $\rightsquigarrow$ $[p \gg 0]$ $X_{\mathbb{F}_p} / \operatorname{Spec}(\mathbb{F}_p)$

P.b. $\mathcal{P}_{\mathbb{F}_p}$ on $X_{\mathbb{F}_p}$ $\Rightarrow$ [no self ext-s] $\hat{\mathcal{P}}$ - P.b. on f-l neigh-d of $X_{\mathbb{F}_p}$ in X_R - f-l sch/$R^{\wedge_p}$

$\rightsquigarrow$ [$\mathbb{G}_m$-equivariance] $\rightsquigarrow$ $\mathcal{P}_{R^{\wedge_p}}$ on $X_{R^{\wedge_p}}$ ($R^{\wedge_p}$ - p-adic numbers)

can be def over alg. ext-n of $\mathbb{Q}$ $\rightsquigarrow$ $\mathcal{P}$ on X

So work w. $X/\mathbb{F}_p$ = $X/\overline{\mathbb{F}}_p$ + Frobenius $Fr: X \longrightarrow X^{(1)}$ - Frobenius twist. ($X^{(1)} \simeq X$)

Nice rep.th. feature of char p: usual non-comm. alg-s have large center

e.g. V - sympl. vect. space $\rightsquigarrow$ $W(V)$ - Weyl alg. ($W(V) = \mathcal{D}(U)$, $U < V$ - lagrang.)

$v \in V \Rightarrow v^p \in$ center $W(V)$, center $W(V) = S(V^{(1)})$. same space w. modif mult-n

Def: Y - alg. scheme; Azumaya alg. $\mathcal{A}$ on Y = sheaf of alg-s triv. in flat topol

i.e $\exists$ fully faithf $\tilde{Y} \xrightarrow{f} Y$ s.t. $f^*\mathcal{A} = Mat_n(\mathcal{O}_{\tilde{Y}})$

Example: $W(V)$ - Azumaya / $V^{(1)}$ (hint: $\sqrt[p]{\cdot}$)

Non-example: $W(V)^{\Gamma_n}$ - not Azumaya / $V^{(1)}/\Gamma_n$ ($V = \overline{\mathbb{F}}_p^{2n}$)

Informal: Azumaya $\Leftrightarrow$ center is functions on symp-c var-ty

Idea (R.B.-Kaledin) Microloc. q-n, $\mathcal{A}/X$, Azumaya/$X^{(1)}$ s.t. $\Gamma(\mathcal{A}) = W(V)^{\Gamma_n}$

(alg-s / $S(V^{(1)})^{\Gamma_n}$) $\Downarrow$ $H^i(\mathcal{A}) = 0$, $i > 0$

Thm: $\exists \mathcal{A}$.

Appl-n to P.b:

$$\begin{tikzcd}
\mathcal{L} \otimes_{\mathcal{O}_{X^{(1)}}} \bullet & D^b(\mathcal{A}) \arrow[r, "R\Gamma", "\sim"'] & D^b(W(V)^{\Gamma_n}) \arrow[r, "W(V)\otimes_{W(V)^{\Gamma}} \bullet", "\sim \; p\gg 0"'] & D^b(W(V)\#\Gamma_n) \arrow[d, "?"', "S"] \\
& D^b(\mathcal{O}_{X^{(1)}}) \arrow[u, "\sim ?"] & & D^b(S(V^{(1)})\#\Gamma_n)
\end{tikzcd}$$

$\exists$ if $\mathcal{A}$ splits: $\mathcal{A} = End(\text{vect. bd-le})/\mathcal{L}$

$\exists$ $W(V)$ splits Γ_n-equiv

FALSE

But true/formal neig-ds of $\pi^{-1}(0)$ in $X^{(1)}$, 0 in $V^{(1)}$

$\rightsquigarrow$ $D^b(\mathcal{O}_{\hat{X}^{(1)}}) \xrightarrow{\sim} D^b(S(\widehat{V^{(1)}})\#\Gamma_n)$ call $\hat{X}^{(1)}$ call $\hat{V}^{(1)}$

$\hat{\mathcal{P}} \longleftrightarrow S(\widehat{V^{(1)}})\#\Gamma_n$

P.b.!, ext-ds to X by $\mathbb{G}_m$-equiv.

3) $\mathbb{H} = T(\mathbb{C}^{2n})\#\Gamma_n[h,\kappa,c_1\dots c_r]/[u,v] = h\omega(u,v)+\sum_{i=0}^{r} c_i \sum_{s\in S_i}\omega_s(u,v)s$ $(c_0=\kappa)$
- univ. grad. flat deform-n of $S(\mathbb{C}^{2n})\#\Gamma_n$; $H_{\cdot,\cdot}$ - spec-n
$\mathcal{A}=[D_h(R)/D_h(R)\Phi([\mathfrak{g},\mathfrak{g}])]^G$ - alg. / $S(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])[h]$, $\mathcal{A}_\lambda(v,w)$ - spec-n w. $h=1$.
Want: isom. $\mathcal{A}\xrightarrow{\sim} e\mathbb{H}e$ of grad. alg-s, lin-r on param-s (w. f-la of Lees) $(h\mapsto h)$
Problem: $\mathcal{A}$, $e\mathbb{H}e$ - not univ.!
Fix: use P.6: $\mathcal{D}=[D_h(R)/D_h(R)\Phi([\mathfrak{g},\mathfrak{g}])|_{(T^*R)^{\theta\text{-}ss}}]^G$ - sheaf on X
w. $\Gamma(\mathcal{D})=\mathcal{A}^{\wedge_h}$
& $\mathcal{D}/(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]\oplus\mathbb{C}h)=\mathcal{O}_X$
$Ext^1(\mathcal{P},\mathcal{P})=0$ & $\mathbb{C}^\times$-equiv $\Longrightarrow$ $\mathcal{P}$ ext-s to right $\mathcal{D}$-module $\tilde{\mathcal{P}}_h$
$End_{\mathcal{D}}(\tilde{\mathcal{P}}_h)$ - deforms $End(\mathcal{P})$, has triv. cohomol. w $i>0$,
$\Gamma(End_{\mathcal{D}}(\tilde{\mathcal{P}}_h))=End_{\mathcal{D}}(\tilde{\mathcal{P}}_h)$ - deforms $End_{\mathcal{O}_X}(\mathcal{P})=S(\mathbb{C}^{2n})\#\Gamma_n$, graded, flat
$\Rightarrow \exists!$ lin. map $Span(h,\kappa,c_1\dots c_r)\xrightarrow{\nu=\nu_{\mathcal{P}}}\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]\oplus\mathbb{C}h$ s.t.
$$End_{\mathcal{D}}(\tilde{\mathcal{P}}_h)=S(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])[[h]]\otimes_{\mathbb{C}[h,\kappa,c_1..c_r]}\mathbb{H}$$

$\mathcal{P}\curvearrowleft\Gamma_n\rightsquigarrow\Gamma_n\to End_{\mathcal{D}}(\tilde{\mathcal{P}}_h)\rightsquigarrow e\,End_{\mathcal{D}}(\tilde{\mathcal{P}}_h)e=End_{\mathcal{D}}(e\tilde{\mathcal{P}}_h)=[e\tilde{\mathcal{P}}_h$ deforms $e\mathcal{P}$ $=\mathcal{O}_X$; uniq. of def-s$]=End_{\mathcal{D}}(\mathcal{D})=\mathcal{A}$.
$$\mathcal{A}=S(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])[h]\otimes_{\mathbb{C}[h,\kappa,c_1..c_r]}e\mathbb{H}e$$

Q: how to recover ν: $\cdot h\mapsto h$ (comp. Poisson $\{\cdot,\cdot\}$-s on $S(\mathbb{C}^{2n})^{\Gamma_n}$)
- $n=1$ - easier.
- gen-l n: reduction to Γ_1 (controls $c_1\dots c_r$) & $\mathbb{Z}/2\mathbb{Z}$ (controls κ)

via taking completions at pts of leaves of codim 2 in $\mathbb{C}^{2n}/\Gamma_n$
$\rightsquigarrow$ ν up to action of $\underbrace{W}_{\text{from quant. LMN isom-s}}\times\mathbb{Z}/2\mathbb{Z}$ $\longleftarrow$ act by isom-s of $\mathcal{A}/e\mathbb{H}e$
+Thm: $W\times\mathbb{Z}/2\mathbb{Z}$ is gr-p of grad. autom-s of $\mathcal{A}$, $=id$ on $S(\mathbb{C}^{2n})^{\Gamma_n}$ & pres-g $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]\oplus\mathbb{C}h$

~~7.4) RHom(~~

Rem: $h=0$: $\mathcal{A}_{0,\lambda} = \mathbb{C}[\mu^{-1}(\lambda)]^G$ - commut.

$\tilde{\mathcal{P}}$ - ext-n of $\mathcal{P}$ to $\tilde{X} = \mu^{-1}(\mathfrak{g}^{*G})^{\theta-ss}/G$, $\mathbb{H}/(h) = End_{\mathcal{O}_{\tilde{X}}}(\tilde{\mathcal{P}})$.

- $\Rightarrow$ [Satake isom: holds $\forall$ SRA] center $\mathbb{H}/(h) \xrightarrow{z \mapsto ez} e\mathbb{H}e/(h)$
- ~~λ gen-c $\Rightarrow \tilde{\mathcal{P}}_\lambda = \mathcal{H}_{0,c}e$~~ $\Rightarrow \Gamma(\tilde{\mathcal{P}}) = \mathbb{H}e/(h)$.

7.4) $R\mathrm{Hom}(\tilde{\mathcal{P}}_h, \bullet): D^b(\mathcal{U}^{opp}\text{-mod}) \xrightarrow{\sim} D^b(\mathbb{H}^{opp}\text{-mod})$ b/c true w/o deform-n

$\rightsquigarrow D^b(\mathcal{A}^\theta_\lambda(n\delta,\epsilon_0)\text{-mod}) \xrightarrow{\sim} D^b(H_c\text{-mod})$ - version of derived loc-n

$\cup \qquad\qquad\qquad\qquad \cup$

$D^b_{\pi^{-1}(0)}(\mathcal{A}^\theta_\lambda(n\delta,\epsilon_0)\text{-mod}) \xrightarrow{\sim} D^b_{fin}(H_c\text{-mod})$

So quiv. var-ty RE conj on # fin. dim. irreps $\Rightarrow$ orig. conj.

Another appl.: $\Gamma_1 = \mathbb{Z}/\ell\mathbb{Z}$, c, c' s.t. $\lambda - \lambda' \in \mathbb{Z}^{Q_0} \Rightarrow$

$D^b(\mathcal{O}_c) \xrightarrow{\sim} D^b(\mathcal{O}_{c'})$ (spec. case of Rouquier's conj.)

7.5)

a) Classif. of P.b:

$\mathcal{P}_1, \mathcal{P}_2$ - P.b. $\rightsquigarrow \nu_{\mathcal{P}_1}, \nu_{\mathcal{P}_2}$: $Span(h, K, c_1, \dots c_r) \longrightarrow \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \oplus \mathbb{C}h$

$\nu_{\mathcal{P}_1} = \nu_{\mathcal{P}_2} \Rightarrow \Gamma(\tilde{\mathcal{P}}_1) = \Gamma(\tilde{\mathcal{P}}_2)$ (as $\mathbb{C}[\tilde{X}]$-mod-s) [$\pi: \tilde{X} \to Spec(\mathbb{C}[\tilde{X}])$

- isom. in codim 2] $\Rightarrow \tilde{\mathcal{P}}_1 \simeq \tilde{\mathcal{P}}_2$ in codim 2 $\Rightarrow \tilde{\mathcal{P}}_1 \simeq \tilde{\mathcal{P}}_2 \Rightarrow \mathcal{P}_1 \simeq \mathcal{P}_2$

So $\#\{P.b\} \leq \#\{\nu\} = 2|W|$

R.B.-Kaledin $\Rightarrow$: Az. alg. $\mathcal{A} \rightsquigarrow$ P.b;

$2|W|$ options for $\mathcal{A}$: $\mathcal{A} = \mathcal{A}^\theta_\lambda(n\delta,\epsilon_0)$, λ - $W\times\mathbb{Z}/2\mathbb{Z}$-conj to $\lambda_0 \leftrightarrow c=0$.

$\hookrightarrow 2|W|$ diff. P.b

R.B. & Finkelberg

b) Macdonald positivity: $\Gamma_1 = \mathbb{Z}/\ell\mathbb{Z}$ (thang. prop-s for $\mathcal{P}/\mathfrak{h}\mathcal{P}$, $\mathcal{P}/\mathfrak{h}^*\mathcal{P}$, $\mathfrak{h}, \mathfrak{h}^* \in S(\mathbb{C}^\ell)^{\# \Gamma_n}$ supports of $= End(\mathcal{P})$).

$\rightsquigarrow$ Verma $\Delta(\lambda)/\mathbb{H}$

under equiv. in 7.4

Q-n: im of $\Delta(\lambda)$ in $D^b(\mathcal{U}\text{-mod})$; $\Delta(\lambda) = \mathbb{H}/\mathbb{H}\mathfrak{h} \otimes_{\Gamma_n} \lambda^*$

A: $R\Gamma(\mathcal{E}nd_{\mathcal{U}}(\tilde{\mathcal{P}}_h)) = \mathbb{H}$

Constr-n of $\mathcal{P} \Rightarrow End(\mathcal{P})$ is flat / $S(\mathfrak{h}), S(\mathfrak{h}^*)$ - as right module

7.4

Reason: in char p: $\mathcal{E}nd(\mathcal{P})$ is Morita equiv to $\mathcal{A}$-deform-n of $\mathcal{O}$

- flat / $S(\mathfrak{h})^{\Gamma_n}$, $S(\mathfrak{h}^*)^{\Gamma_n}$; then lift to char 0.

$R\Gamma(\mathcal{E}nd(\tilde{\mathcal{P}}_h)/\mathcal{E}nd(\tilde{\mathcal{P}}_h)\mathfrak{h}) = \mathbb{H}/\mathbb{H}\mathfrak{h}$.

So $\Delta(\lambda) \mapsto \mathcal{E}nd(\tilde{\mathcal{P}}_h)/\mathcal{E}nd(\tilde{\mathcal{P}}_h)\mathfrak{h} \otimes_{\Gamma_n} \lambda^* \xmapsto{e\cdot} \tilde{\mathcal{P}}_h^*/\tilde{\mathcal{P}}_h^*\mathfrak{h} \otimes_{\Gamma_n} \lambda^* \in \mathcal{A}$-mod

Cor: im $\Delta(\lambda)$ is object.

+ $\Delta(\lambda)$ supposed to have upper triang. prop-ty:

Set $h=0$: $\Delta_0(\lambda) \mapsto \tilde{\mathcal{P}}^*/\tilde{\mathcal{P}}^*\mathfrak{h} \otimes_{\Gamma_n} \lambda^*$ - sheaf / $\tilde{X}$-scheme / $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]^*$

Hamilt. $\mathbb{C}^\times \curvearrowright \tilde{X}$ (from $\mathbb{C}^\times \curvearrowright R \oplus R^*$, $t\cdot(r,\alpha) = (tr, t^{-1}\alpha)$) - fin. many fixed pts

on $\forall$ fiber / $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. $\xleftrightarrow{(*)}$ ℓ-part-ns on n

$\lambda \rightsquigarrow L_\alpha(\lambda)$ - contract. comp in $\tilde{X}_\alpha$, $\alpha \in (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*$

α-gen-c $\tilde{\mathcal{P}}_\alpha^*/\tilde{\mathcal{P}}_\alpha^*\mathfrak{h} \otimes_{\Gamma_n} \lambda^* = \mathcal{O}_{L_\alpha(\lambda)}$ (includes def-n of $(*)$)

$\alpha \to 0$: supp of $\boxed{\mathcal{P}^*/\mathcal{P}^*\mathfrak{h} \otimes_{\Gamma_n} \lambda^* \subset \bigcup_{\mu \le \lambda} L_\alpha(\mu)}$ - important concl-n

geom. order

$\mu \le \lambda$: $\overline{L_0(\lambda)} \cap L_0(\mu) \neq \emptyset$ ($\ell=1$: dominance order)

~~Concl-n~~: ~~Su~~ Similar: $\mathcal{P}^*/\mathcal{P}^*\mathfrak{h}^* \otimes_{\Gamma_n} \lambda^* \subset \bigcup_{\mu \ge \lambda} L_\alpha(\mu)$

$\mathcal{P}$ w. $\nu_{\mathcal{P}}$ as in lect. 4 $\rightsquigarrow$ wreath Macdonald positivity.

($\ell=1$) usual Macdonald positivity.

Lecture 9. Schur-Weyl dualities & Rouquier's equiv. thm

9.1) Rep. th. of GL_m in char >0
9.2) Classic. Schur-Weyl duality
9.3) Quantum Schur-Weyl duality
9.4) Equiv. thm

9.1) Hyperalgebra: ($\mathbb{K}=\overline{\mathbb{K}}$, ~~char $\mathbb{K}=p>0$~~)

Char 0: Rat-l rep. of GL_m = fin. dim rep. of $\mathcal{U}(\mathfrak{gl}_m)$ s.t. $\mathcal{U}(\mathfrak{t})$ ($\mathfrak{t}=\{diag(x_1 \dots x_m)\}$) acts diag-y w. wts $\in \mathbb{Z}^m$.

Char p: false (e.g. b/c $\mathcal{U}(\mathfrak{gl}_m)$ has large center)

~~Rem: T still acts diag-ly:~~

$\mathcal{U}(\mathfrak{gl}_m) \rightsquigarrow$ hyperalgebra $\dot{\mathcal{U}}(\mathfrak{gl}_m)$ - (w. divided powers)

$\mathcal{U}(\mathfrak{gl}_m(\mathbb{Q})) \supset \dot{\mathcal{U}}_{\mathbb{Z}}$ - gen-d by $\frac{E_{ij}^n}{n!} =: E_{ij}^{(n)}$ $(i\neq j)$ & $\binom{E_{ii}}{n} = \frac{E_{ii}(E_{ii}-1)\dots(E_{ii}+1-n)}{n!}$

$\dot{\mathcal{U}}(\mathfrak{gl}_m) = \mathbb{K}\otimes_{\mathbb{Z}} \dot{\mathcal{U}}_{\mathbb{Z}}$ - not Noeth-n

Rat-l rep-n of $GL_m \overset{(*)}{=}$ fin dim $\dot{\mathcal{U}}(\mathfrak{gl}_m)$-mod s.t. $\dot{\mathcal{U}}(\mathfrak{t})$ acts diagon-ly w. integral wts (reason: T acts diag-ly, char-s of $T=\mathbb{Z}^m$):

$M = \bigoplus_{\mu\in\mathbb{Z}^n} M_\mu$ $\quad\Big|\ \mu\in\mathbb{Z}^n \rightsquigarrow M_\mu = \{m\in M \mid \binom{E_{ii}}{n}m = \binom{\mu_i}{n}m\}$

Idea of proof of $(*)$: $exp(tE_{ij}) = \sum_k t^k E_{ij}^{(k)}$ $(i\neq j)$

Weyl modules: $\lambda\in\mathbb{Z}^m$, $\lambda_1\geqslant\dots\geqslant\lambda_m \rightsquigarrow W(\lambda)\in\dot{\mathcal{U}}(\mathfrak{gl}_m)$

gen-d by v_λ w. rel-ns: $\binom{E_{ii}}{n}v_\lambda = \binom{\lambda_i}{n}v_\lambda \quad \forall n$

$E_{ij}^{(n)}v_\lambda = 0,\ i<j,\ \forall n>0$

$E_{ij}^{(n)}v_\lambda = 0,\ i>j \quad \forall n>\lambda_i-\lambda_j$

(class. rel-ns for high. vector of fin. dim irrep in char 0)

char. $W(\lambda)$ - Weyl f-la

Fact: $Rep(GL_m)$ - high. wt. cat-y w. standards $W(\lambda)$ & usual order on wts

Fact: $\mu\in\mathbb{Z}^m \rightsquigarrow \mathcal{O}(\mu)$ - line bundle on G/B

not used: $W(\lambda) = H^0(\mathcal{O}(-w_0\lambda))^*$, $-w_0\lambda = (-\lambda_m, -\lambda_{m-1},\dots,-\lambda_1)$

9.1

9.2) Def: $V \in Rep(GL_m)$ is polyn of deg n if $V_\lambda \neq 0 \Rightarrow \lambda_i \geq 0\ \forall i$ & $\sum \lambda_i = n$; not-n $Pol^n(GL_m)$; Example $(\mathbb{K}^m)^{\otimes n} \in Pol^n(GL_m)$, all subquot-s, ext-s

$Pol^n(GL_m) \longrightarrow Pol^n(GL_{m-1})$, $V \mapsto V^{T_0}$, $T_0 = \{diag(1,..1,x_m)\}$

Fact: $\xrightarrow{\sim}$ if $m > n$. ; e.g. $(\mathbb{K}^m)^{\otimes n} \mapsto (\mathbb{K}^{m-1})^{\otimes n}$

$GL_m \curvearrowright (\mathbb{K}^m)^{\otimes n} \curvearrowleft \mathfrak{S}_n$; $m \geq n \Rightarrow \mathbb{K}\mathfrak{S}_n = End_{GL_m}((\mathbb{K}^m)^{\otimes n})^{opp}$

char 0: im of $\mathbb{K}GL_m = End_{\mathfrak{S}_n}((\mathbb{K}^m)^{\otimes n})$, all reps are s/s,

$(\mathbb{K}^m)^{\otimes n} = \bigoplus_{\lambda \vdash n} W(\lambda) \otimes S(\lambda)$, $S(\lambda)$ - $\mathfrak{S}_n$-irrep

char p: Fact: $(\mathbb{K}^m)^{\otimes n}$-proj-ve $\rightsquigarrow$ exact $Sh_m = Hom_{GL_m}((\mathbb{K}^m)^{\otimes n}, \bullet) : Pol^n(GL_m)$
$\longrightarrow \mathfrak{S}_n$-mod intertw-d by $Pol^n(GL_m) \xrightarrow{\sim} Pol^n(GL_{m-1})$

Centr. prop. $\Rightarrow Sh_m$ is essent. surj, i.e. quot-t functor.

~~Schur alg-s~~: • $W(\lambda) \in Pol^n(GL_m)$, $\lambda \vdash n$; span $Pol^n(GL_m)$

$\{\lambda \vdash n\}$ -poset ideal in high-t wt poset $\Rightarrow Pol^n(GL_m)$ - high wt cat-y
w. standards $W(\lambda)$

• Schur alg-s: im $U(\mathfrak{gl}_m)$ in $End_{\mathfrak{S}_n}((\mathbb{K}^m)^{\otimes n})$
$=$ quot. of $U(\mathfrak{gl}_m)$ acting on all $V \in Pol^n(GL_m)$

Not-n $S_m(n)$ or $S(n)$ if m isn't important (well-def / Morita equiv.,
i.e $S(n)$-mod is same): $Sh: S(n)\text{-mod} \longrightarrow \mathfrak{S}_n\text{-mod}$

Facts • Sh is fully faithf. on proj-s

• $\bullet \otimes \bullet : S(n_1)\text{-mod} \boxtimes S(n_2)\text{-mod} \longrightarrow S(n_1+n_2)\text{-mod}$
has biadj-t.

Side rem-k: Compare w. $KZ: \mathcal{O}_c(\mathfrak{S}_n) \longrightarrow \mathcal{H}_q(n)\text{-mod}$

$Ind: \mathcal{O}_c(\mathfrak{S}_{n_1}) \boxtimes \mathcal{O}_c(\mathfrak{S}_{n_2}) \longrightarrow \mathcal{O}_c(\mathfrak{S}_{n_1+n_2})$

9.4) Equiv thm (Rouquier) $\mathcal{O}_c(\mathfrak{S}_n) \xrightarrow{\sim} S_q(n)\text{-mod}$, $\Delta(\lambda) \mapsto W(\lambda)$

Proof: Step 1: charact. $P(\lambda) \in \mathcal{O}_c(\mathfrak{S}_n)$

$\mathrm{Ind}^{\mathcal{O}}(\lambda) = \mathrm{Ind}_{\mathfrak{S}_{\lambda_1} \times \ldots \times \mathfrak{S}_{\lambda_k}}^{\mathfrak{S}_n} \Delta(\lambda_1) \boxtimes \ldots \boxtimes \Delta(\lambda_k)$ - proj-ve (Ind. proj-ve $\to$ proj-ve + λ_i is max. part-n of $\lambda_i \Rightarrow \Delta(\lambda_i) \in \mathcal{O}_c(\mathfrak{S}_{\lambda_i})$ is proj-ve)

Stand. subq-s of $\mathrm{Ind}^{\mathcal{O}}(\lambda)$: same as irreps in $\mathrm{Ind}\ \lambda_1 \boxtimes \ldots \boxtimes \lambda_k$

row of λ_1; add λ_2 no two in same column, same w. λ_3 boxes etc.

$\Rightarrow \Delta(\lambda)$ is min. stand in $\mathrm{Ind}^{\mathcal{O}}(\lambda) \Rightarrow \mathrm{Ind}^{\mathcal{O}}(\lambda) \twoheadrightarrow \Delta(\lambda) \Rightarrow P(\lambda)$ is summand in $\mathrm{Ind}^{\mathcal{O}}(\lambda)$ w. mult. 1; uniq. w. prop. that $P(\lambda)$ not summ of $\mathrm{Ind}^{\mathcal{O}}(\tilde{\lambda})$, $\tilde{\lambda} > \lambda$

Step 2: charact. $KZ(P(\lambda))$: uniq. summ of $\mathrm{Ind}^{\mathcal{H}}\lambda :=$ $\mathrm{Ind}_{\mathcal{H}_q(\lambda_1) \boxtimes \ldots \boxtimes \mathcal{H}_q(\lambda_k)}^{\mathcal{H}_q(n)} \mathrm{triv}_{\lambda_1} \boxtimes \ldots \boxtimes \mathrm{triv}_{\lambda_k}$ (triv: all T_i act by q^2)

that doesn't occur in $\mathrm{Ind}^{\mathcal{H}}\tilde{\lambda}$, $\tilde{\lambda} > \lambda$

Step 3: Schur alg.: same! $\Rightarrow KZ(P(\lambda)) = Sh(P^S(\lambda))$

Step 4: KZ & Sh fully faithf. on proj-s: $\Rightarrow$

$\mathcal{O}_c(\mathfrak{S}_n) \simeq \mathrm{End}_{\mathcal{O}}(\bigoplus_{\lambda \vdash n} P(\lambda))^{opp}\text{-mod} \simeq \mathrm{End}_{\mathcal{H}_q}(\bigoplus_{\lambda \vdash n} KZ(P(\lambda)))^{opp}\text{-mod}$ $\simeq S_q(n)\text{-mod}$