Lecture 5.

5.2) Etingof type conjecture for quantized quiver varieties

Quiver Q, $v,w\in \mathbb{Z}^{Q_0}_{\geq 0}$, $\theta\in\mathbb{Z}^{Q_0}$, $\lambda\in\mathbb{C}^{Q_0}$ $\rightsquigarrow$ quiver var-ty $\mathcal{M}^\theta(v,w)$, quant-n

$\mathcal{A}^\theta_\lambda(v,w)$, $\mathcal{A}_\lambda(v,w):=\Gamma(\mathcal{A}^\theta_\lambda(v,w))$

Q: $\#\, \mathrm{Irr}(\mathcal{A}_\lambda(v,w))_{fin}=?$

Assume: homol. dim $\mathcal{A}_\lambda(v,w)<\infty \iff R\Gamma: D^b(\mathcal{A}^\theta_\lambda(v,w)) \xrightarrow{\sim} D^b(\mathcal{A}_\lambda(v,w))$: $L\mathrm{Loc}$

$K_0(D^b_{fin}(\mathcal{A}_\lambda(v,w)))$ - complexes w. fin dim homol.

denote the map by CC_v $\longrightarrow$ $M\mapsto CC(L\mathrm{Loc}\, M)$

$L\mathrm{Loc}: D^b_{fin}(\mathcal{A}_\lambda(v,w)) \longrightarrow D^b_{\pi^{-1}(0)}(\mathcal{A}^\theta_\lambda(v,w))$-mod

homol. have Supp on $\pi^{-1}(0)$ - lagr. subvar

span of comp-s of $\pi^{-1}(0)$ $\underset{\text{Nakajima}}{=}$ $L_\omega[\nu]$

Fact: CC_v is inj-ve.

Q: $\mathrm{Im}\, CC$, $CC:=\bigoplus_v CC_v$
$\cap L_\omega$

Subalg $\mathfrak{a}\subset\mathfrak{g}(Q)$ gen-d by $\mathfrak{h}$ & $\mathfrak{g}(Q)_\beta$, where $\beta=\sum_{i\in Q_0} b_i\alpha^i$ is real root w. $\sum b_i\lambda_i\in\mathbb{Z}$

$L^{\mathfrak{a}}_\omega:=\bigoplus_{\sigma\in W(Q)} U(\mathfrak{a})L_\omega[\sigma\omega]$

Conj (I.L & R.B) $\mathrm{Im}\, CC=L^{\mathfrak{a}}_\omega$

Easier: $L^{\mathfrak{a}}_\omega\subset \mathrm{Im}\, CC$

- $\mathcal{M}^\theta(\sigma\cdot v,w)=pt \Rightarrow \mathcal{A}_\lambda(\sigma\cdot v,w)=\mathbb{C}$ - has uniq. irrep, fin dim-l

$\beta\rightsquigarrow$ Nakajima op-rs e_β, f_β on L_ω.

Want: endofunctors E_β, F_β of $\bigoplus_v D^b(\mathcal{A}^\theta_\lambda(v,w))$

preserving $\bigoplus_v D^b_{\pi^{-1}(0)}(\ldots)$

w. $e_\beta\circ CC=CC\circ E_\beta$, $f_\beta\circ CC=CC\circ F_\beta$.

Harder: $\mathrm{Im}\, CC\subset L^{\mathfrak{a}}_\omega$.

Lecture 10. Categorical Kac-Moody actions

1) Motivation/example: cyclotomic HA & RCA.
2) General def-n
3) Crystal.
4) Rickard complex
5) Structure theory (?)

10.1.1) RCA $H_{1,c}(n)$ for $G(\ell,1,n) \rightsquigarrow$ cat-y $\mathcal{O}_c(n) \rightsquigarrow \mathcal{O}_c = \bigoplus_{n=0}^{+\infty} \mathcal{O}_c(n)$ ($\mathcal{O}_c(0) = Vect$)

$M \in Irr(\mathcal{O}_c) \rightsquigarrow Supp\, M = W\{(0,..0, y_1...y_1, x_1...x_1, x_2...x_2, ... x_j...x_j)$

$E = \bigoplus_{n=0}^{+\infty} Res_n^{n-1}$, $Res_n^{n-1}: \mathcal{O}_c(n) \longrightarrow \mathcal{O}_c(n-1)$ assoc. to $G(\ell,1,n-1) \subset G(\ell,1,n)$

$E^k M = 0\ \forall k > i \Rightarrow [M] \in Ker[E]^k$ (level of K_0) - very little info

Improve: $E = \bigoplus_{z\in\mathbb{C}} E_z \Rightarrow [M] = \bigcap Ker[E_{z_1}]...[E_{z_k}]$.

Goal: produce + compute $[E_z]$.

10.1.2) Decomp-n $c \rightsquigarrow$ param-s (q, Q), $Q = (Q_1, ... Q_\ell)$ - param-s for cyclot. HA

$\mathcal{H}_{q,Q}$-mod $= \bigoplus_{n\geq 0} \mathcal{H}_{q,Q}(n)$-mod $\rightsquigarrow$ endof-r ${}^{\mathcal{H}}E$ s.t. $\underline{KZ \circ E = {}^{\mathcal{H}}E \circ KZ}$

$(KZ = \bigoplus_{n\geq 0} KZ_n)$; KZ is f.faith. on $\mathcal{O}_c$-proj $\Longrightarrow$ $\Downarrow$

has biadj. $\leftarrow E: \mathcal{O}_c\text{-proj} \to \mathcal{O}_c\text{-proj} \Longrightarrow End(E) = End({}^{\mathcal{H}}E)$

so $\underbrace{{}^{\mathcal{H}}E = \bigoplus {}^{\mathcal{H}}E_z}_{z} \rightsquigarrow E = \bigoplus_z E_z$

use endom. $X \in End({}^{\mathcal{H}}E)$: $X_n \in \mathcal{H}_{q,Q}(n)$ commutes w. $\mathcal{H}_{q,Q}(n-1) \hookrightarrow \mathcal{H}_{q,Q}(n)$

$\rightsquigarrow$ mult-n by $X_n \in End({}^{\mathcal{H}}Res_n^{n-1}) \rightsquigarrow X = \bigoplus_n X_n$

$\rightsquigarrow {}^{\mathcal{H}}E_z :=$ gener. eigenfunctor in ${}^{\mathcal{H}}E$ w. e-value z

$({}^{\mathcal{H}}E_z M =$ gen. e-space for X_n w. e-value z, $M \in \mathcal{H}_{q,Q}(n)$-mod$)$

10.1.3) $[E_z]$

a) q, Q - generic ($q \neq \sqrt[\cdot]{1}$, $Q_i/Q_j \notin q^{\mathbb{Z}}$)

Rep. th. of $\mathcal{H}_{q,Q}(n)$ - same as of $G(\ell,1,n)$ - remf. of $\mathfrak{S}_n$-case

• s/simple w. simples L_λ, λ - ℓ-mult-n of n

• basis $v_T \in L_\lambda$, T - ℓ-multitableau on $\{1,2,\dots,n\}$, e.g. $\begin{array}{|c|c}\hline 3 \\ \hline 1 & \multicolumn{1}{c|}{4} \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 2 & 5 \\ \hline \end{array}$

• $X_k v_T = Q_i q^{x-y}$, if box № k in T is in ith part-n, column x, row y.

So ${}^{\mathcal{H}}E_z L_\lambda = \begin{cases} L_\mu, & \lambda = \mu \sqcup \text{box}(x,y,i) \text{ w. } z = Q_i q^{x-y} \\ 0, & \text{else} \end{cases}$ ← all diff. numbers

Gen-ℓ case: $c \rightsquigarrow$ curve $c(t)$ w. $c(0)=c$, $c(t)$ s.t q, Q gen-c, $t \neq 0$

$[\mathcal{O}_c]$ indep. of c; $\Delta(\lambda)$, E, X - cont. in c;

at $t\neq 0$ e-values $z_1(t), \dots, z_k(t)$ - all diff-t

$\Rightarrow {}^{\mathcal{H}}E_z$ is deg-n to $t=0$ of $\bigoplus_{i, \lim z_i(t) = z} E_{z_i(t)}$

$\Rightarrow [E_z][\Delta_c(\lambda)] = \sum_\mu [\Delta_c(\mu)]$, over all μ s.t $\lambda = \mu \sqcup \text{box}(x,y,i)$

$z = Q_i q^{x-y}$

10.1.4) Biadj-t functors

$F = \bigoplus_{n \geq 0} \mathrm{Ind}_n^{n+1} : \mathcal{O}_c \longrightarrow \mathcal{O}_c$, ${}^{\mathcal{H}}F$ - biadj. to

E $\qquad {}^{\mathcal{H}}E$

fix adj-s (e.g. $\mathbf{E}$ = left adj to E) $\rightsquigarrow \mathrm{End}(F) \cong \mathrm{End}(E)^{op}$

$\rightsquigarrow F = \bigoplus_z F_z$ w. F_z = left adj to E_z

Claim: $F_z \simeq$ right adj-t of E_z.

Proof: $Z = X_1 \dots X_n \in$ center $\mathcal{H}_{q,Q}(n) \rightsquigarrow Z \in \mathrm{End}(1_{\mathcal{O}_c(n)}) \rightsquigarrow$

decomp-n $\mathcal{O}_c(n) = \bigoplus_{u \in \mathbb{C}^\times} \mathcal{O}_c(n)^u$

$\left.\begin{array}{l} E_z : \mathcal{O}_c(n)^u \longrightarrow \mathcal{O}_c(n)^{uz^{-1}} \\ \Downarrow \\ F_z : \mathcal{O}_c(n)^u \longrightarrow \mathcal{O}_c(n)^{uz} \end{array}\right\}$ $\bigoplus_z F_z$ - right adj to $\bigoplus_z E_z \Rightarrow$

$F_z \simeq$ right adj to E_z.

On K_0: $[F_z][\Delta(\lambda)] = \sum_\nu [\Delta(\nu)]$, $\nu = \lambda \sqcup \text{box}(x,y,i)$ w. $Q_i q^{x-y} = z$.

10.1.5) Spec. choice of param-s: $Q_i = q^{s_i}$ $s_i \in \mathbb{Z}$

$E_i = E_{q^i}$, $F_i = F_{q^i}$ $(i \in \mathbb{Z})$ other $E_z, F_z = 0$.

Claim: op-rs $[E_i], [F_i]$ define action of $\mathfrak{g}$ =

• $\mathfrak{sl}_\infty$, $q \neq \sqrt{1}$, - Lie alg. w. Dynkin diagr 

• $\widehat{\mathfrak{sl}}_e$, q = prim. $\sqrt[e]{1}$ 

Proof: comput-n

$[\mathcal{O}_\ell]$ - level ℓ Fock space w. multicharge $(s_1 \dots s_\ell)$ - h.wt $\sum s_i \omega_i$

In fact, $[\mathcal{H}_{q,Q}\text{-mod}]$ - irrep. w. h.wt $\sum s_i \omega_i$, ω_i - fund wt.

10.1.6 Remarks:

a) Have homom $\mathcal{H}_q^{aff}(m) \longrightarrow \operatorname{End}(\overset{q}{E}{}^m) = \operatorname{End}(E^m)$

b/c $\mathcal{H}_q^{aff}(m) \longrightarrow$ centr-r of $\mathcal{H}_{q,Q}(n-m)$ in $\mathcal{H}_{q,Q}(n)$

Comes from 2 functors $X \in \operatorname{End}(E)$, $T \in \operatorname{End}(E^2)$ - from $T_{n-1} \in \mathcal{H}_{q,Q}(n)$

$X_i \longmapsto 1^{i-1} X 1^{m-i}$, $T_j \longmapsto 1^{j-1} T 1^{n-j-1}$

Also $\mathcal{H}_q^{aff}(m) \longmapsto \operatorname{End}(F^m)$ b/c $\mathcal{H}_q^{aff}(m) = \mathcal{H}_q^{aff}(m)^{opp}$

b) $P \in \mathbb{C}[X_1^{\pm 1}, \dots X_n^{\pm 1}]^{\mathfrak{S}_n} \subset$ center $\mathcal{H}_q^{aff}(n) \rightsquigarrow \mathbb{C}[X_1^{\pm 1}, \dots X_n^{\pm 1}] \longrightarrow \operatorname{End}(1_{\mathcal{O}_\ell(n)})$

$\rightsquigarrow$ decomp-n of $\mathcal{O}_\ell(n)$

Claim: on level of K_0, decomp-n of $\mathcal{O}_\ell(n)$ = wt. decomp of $[\mathcal{O}_\ell]$ w.r.t $\mathfrak{g}$.

Summand is param by $\sum n_i \alpha_i$ (α_i is simple root)

- Serre span of $\Delta(\lambda)$, ~~# boxes~~ s.t.

n_i = # boxes in λ s.t shifted content $\operatorname{cont}(x,y,i) = s_i + x - y$

$= i$ - in $\mathbb{Z}$ if $\mathfrak{g} = \mathfrak{sl}_\infty$

in $\mathbb{Z}/e\mathbb{Z}$ if $\mathfrak{g} = \widehat{\mathfrak{sl}}_e$.

Hint: see what happens for generic q, Q, then degenerate

2)

10.2.1) Actions on abelian cat-s

$\mathcal{C}$: $\mathbb{C}$-lin artinian abel. cat-y w. enough proj-s

Data: $q \in \mathbb{C}^\times \setminus \{1\}$, functors $E, F: \mathcal{C} \to \mathcal{C}$, $X \in End(E)$, $T \in End(E^2)$

Axioms:

1) E, F - biadj-t. (usually fix $1 \to FE$, $EF \to 1$

2) $E = \bigoplus_i E_i$, E_i := gen-d e-functor for X w. e-value q^i

$\rightsquigarrow F = \bigoplus_i F_i$ - from left adj-s

3) funct-s E_i, F_i define $\mathfrak{g} = \mathfrak{gl}_\infty$ ($q \neq \sqrt{1}$) or $\hat{\mathfrak{sl}}_e$ (q = prim $\sqrt[e]{1}$) action on $[\mathcal{C}]$ s.t. $[\mathcal{C}]$ is integr-le & $\mathcal{C} = \bigoplus_\mu \mathcal{C}_\mu$ w. $[\mathcal{C}_\mu]$ = = wt. space of wt. μ

4) $X_i \mapsto 1_E^{i-1} X 1_E^{n-i}$, $T_j \mapsto 1_E^{j-1} T 1_E^{n-j-1}$ defines homom $\mathcal{H}_q^{aff}(n) \to End(E^n)$ $\forall n$.

Examples: $\mathcal{O}_c$, $\mathcal{H}_{q,Q}$-mod

Rem: Other types A - see below (*)

Rem: Same works w. $\mathbb{C}$-lin. cat-s (replace $[\mathcal{C}]$ w. split Grothendieck grp)

- more gen-l setting: abelian $\mathcal{C} \rightsquigarrow \mathbb{C}$-lin. $\mathcal{C}$-proj

Rem: can use other versions of aff HA: degenerate AHA, nil. AHA etc

10.2.2) Kac-Moody 2-algebra

Above def-n - "naive": parallel def-n from ordin rep-n th: "action of algebra" = collection of operators". Want an analog of def-n of module/algebra:

module $\rightsquigarrow$ category; algebra = ~~monoid~~ cat-y w. single object $\rightsquigarrow$ 2-category

Example: $\mathfrak{sl}_2$: 2-cat-y categorifying "idemp ~~ver~~ version" of $U(\mathfrak{sl}_2)$

- add projectors 1_n to comp-t of wt n but remove h

- basis $1_m E^n F^k 1_{m+2(k-n)}$, $n, k \geq 0$, $m \in \mathbb{Z}$

- alg-ra w/o unit acting on wt rep-ns of $\mathfrak{sl}_2$

Categorif-n $\mathcal{U}$: - 2-cat-y

objects: $\mathbb{Z}$

1-morphisms $Mor(n,m)$ - ~~monomials~~ direct sums of monomials in E, F of required wt. (e.g. EF^2E1_m)

2-morphisms, incl. $X \in End(E1_n)$, $T \in End(E^2 1_n)$, $\eta \in Hom(EF1_n, 1_n)$

$\varepsilon: Hom(1_n, FE1_n)$, generating arb. Hom-s & subj to rel-ns)

~~Rep-n of $\mathcal{U}(\mathfrak{sl}_2)$~~

- can improve to the idemp comp-n $\dot{\mathcal{U}}(\mathfrak{sl}_2)$ adding divided powers (see below)
- also have graded version categorifying $\dot{\mathcal{U}}_q(\mathfrak{sl}_2)$.
- generalizes to $\dot{\mathcal{U}}_q(\mathfrak{g})$, $\mathfrak{g}$ - KM algebra

Rep-n of $\dot{\mathcal{U}}(\mathfrak{sl}_2) \rightsquigarrow$ cat-l $\mathfrak{sl}_2$-action in usual sense

from page 4: (*) Can give similar def-ns for cat-l actions of $\mathfrak{sl}_m$

E.g. $\hat{\mathfrak{sl}}_2$: E, F s.t. X has single e-value a on E

& $E, F \rightsquigarrow \mathfrak{sl}_2$-action; cat-l $\hat{\mathfrak{sl}}_e$-action $\rightsquigarrow e$ cat-l $\mathfrak{sl}_2$-actions

3) Crystals

Q: have basis $[L]$, $L \in Irr\, \mathcal{C}$ in $[\mathcal{C}]$. What can we say about it?

10.3.1) E, F-actions on simples - $\mathfrak{sl}_2$-cat-n.

Reminder: $M \in \mathcal{C} \rightsquigarrow head(M) =$ max s/simple quotient

$soc(M) =$ max. s/simple sub

Thm (Chuang-Rouquier) $L \in Irr\, \mathcal{C}$, $n =$ max w. $E^n L \neq 0$

(1) $soc\, EL = head\, EL$ is simple, L'

(2) mult. of L' in $EL = n$

(3) $\forall L'' \neq L'$ - simple subquot. of EL have $E^{n-1} L'' = 0$

Similar for F

10.3.2) Perfect bases & crystals.

V - integr-le rep-n of $\hat{\mathfrak{sl}}_e$

Def: A wt. basis $\mathcal{B} \subset V$ is <u>perfect</u> if $\forall\, i \in \mathbb{Z}/e\mathbb{Z};\ \forall b \in \mathcal{B}$

$e_i b = n_i^+(b) b' + b_0'$, $n_i^+(b)$ - max n s.t. $e_i^n b \neq 0$, $b' \in \mathcal{B}$, b_0' s.t. $e_i^{n-1} b_0' = 0$

$f_i b = n_i^-(b) b'' + b_0''$ - similar.

Example: basis $\{[L],\ L \in \mathrm{Irr}\, \mathcal{C}\}$ in $\mathcal{C}$.

Crystal: collection of maps $\tilde{e}_i, \tilde{f}_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$

$\tilde{e}_i : b \mapsto b'$, $n_i^+(b) > 0$ or $b \mapsto 0$, else

$\tilde{f}_i : b \to b''$ $n_i^-(b) > 0$ or $b \mapsto 0$, else

Thm (Berenstein - Kazhdan) crystals for 2 perfect bases, $\mathcal{B}^1, \mathcal{B}^2$ are isomorphic (and isom to Kashiwara's crystal def. via q. groups)

10.3.3) Applications:

a) $[\mathcal{H}_{q,Q}\text{-mod}]$ is irrep $V(\omega)$, where ω is as follows:

$q = \mathrm{prim}\ \sqrt[e]{1}$, $Q = (q^{s_1}, \dots q^{s_\ell})$, $s_1 \dots s_\ell \in \mathbb{Z}$: $\omega = \omega_{s_1} + \omega_{s_2} + \dots + \omega_{s_\ell}$.

Proof: $[\mathcal{O}_c] \xrightarrow[\,[KZ]\,]{} [\mathcal{H}_{q,Q}\text{-mod}]$, vector $\mathbb{1} = [\Delta(\varnothing)]$ has wt. ω

$\Rightarrow [\mathcal{H}_{q,Q}\text{-mod}]$ has sing. vector of wt. ω. Remains: irreducibility

Perfect basis $[L]$: $\tilde{e}_i L = \mathrm{head}(E_i L)$ so $\tilde{e}_i L = 0\ \forall i \Rightarrow E_i L = 0\ \forall i$

$\Rightarrow \mathrm{Res}_n^{n-1} L = 0 \Rightarrow n = 0$: unique el-t $L \in \mathcal{B}$ killed by $\tilde{e}_i\ \forall i$.

If $[\mathcal{H}_{q,Q}\text{-mod}] = V_1 \oplus V_2$: ~~two~~ perf. basis $\mathcal{B}_1 \sqcup \mathcal{B}_2$, 2 el-ts (highest vectors) annih. by all $\tilde{e}_i$ - contradiction $\square$

b) Etingof's conj: recall $L \in \mathrm{Irr}\, \mathcal{O}_c \rightsquigarrow \mathrm{Supp}\, L \rightsquigarrow i(L) \in \mathbb{Z}_{\geq 0}$.

$i(L) = 0$: $\dim \bigcap_{i \in \mathbb{Z}/e\mathbb{Z}} \mathrm{Ker}\, e_i|_{[\mathcal{O}_c]} = \{b \in \mathcal{B} \mid \tilde{e}_i b = 0\ \forall\, i \in \mathbb{Z}/e\mathbb{Z}\}$

so this space is spanned by classes of simples there $\rightsquigarrow$ can count those;

$i(L) \leq n$ - similar (using prop. of crystals & Ind-functors) $\bigcap \mathrm{Ker}$ (monomials of deg n in e_i's) is spanned by classes of simples.

10.4) Rickard complex

10.4.1) Goal: categorify simple reflections in Weyl groups

$\mathfrak{sl}_2$-cat-n: $\mathfrak{sl}_2 \curvearrowright V \rightsquigarrow SL_2(\mathbb{C}) \curvearrowright V \rightsquigarrow s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \curvearrowright V$

V_α - wt. space of wt $\alpha \in \mathbb{Z}$, $v \in V_\alpha$

$$sv = \sum_{j \geq \max(0,-\alpha)} (-1)^j \frac{e^j}{j!} \frac{f^{j+\alpha}}{(j+\alpha)!}$$

To categorify: • need divided power functors $E^{(j)}, F^{(j+\alpha)}$
• form complex.

10.4.2) Divided powers: $M \in \mathcal{C} \rightsquigarrow E^n M \in H_q^{aff}(n)$-mod s.t. $X_1, \dots X_n$ act w. single e-value, a. In this subcat. of $H_q^{aff}(n)$-mod have only one simple $K_n = \mathrm{Ind}_{\mathbb{C}[X_1^{\pm 1}, \dots X_n^{\pm 1}]}^{H_q^{aff}(n)} \mathbb{C}$ - of dim $n!$

$\Rightarrow E^n M$ <u>funct.</u> decomp into $(E^{(n)} M)^{\oplus n!}$
simil. for $F^n M$

Complex

10.4.3)
$$\begin{array}{ccc} E^{(j)} F^{(j+\alpha)} & \xrightarrow[\text{special direct summand}]{} & E^j F^{j+\alpha} = E^{j-1} E F F^{j+\alpha-1} \\ \downarrow d & & \downarrow \text{adj. morphism} \\ E^{(j-1)} F^{(j+\alpha-1)} & \longrightarrow & E^{j-1} F^{j+\alpha-1} \end{array}$$

- gives complex Θ: $\dots \to E^{(j)} F^{(j+\alpha)} \to E^{(j-1)} F^{(j+\alpha-1)} \dots$ (Chuang-Rouquier)
- defining equiv. $K^b(\mathcal{C}_\alpha) \xrightarrow{\sim} K^b(\mathcal{C}_{-\alpha})$

10.4.4) Geometric example

Goal: equiv. $D^b(\mathrm{Coh}\, T^* Gr(v, w)) \xrightarrow{\sim} D^b(\mathrm{Coh}\, T^* Gr(w-v, w))$.

via cat-l action of $\mathfrak{sl}_2$ on $\bigoplus_v \underbrace{D^b(\mathrm{Coh}\, T^* Gr(v, w))}_{\mathcal{C}_v}$

Lecture 11

Etingof's conj. for cyclotomic cat-s $\mathcal{O}$ (after Shan-Vasserot)

1) Reminder

2) Cat-l Heisenberg action

3) Level-rank duality

11.1) $c \rightsquigarrow (\kappa = \frac{r}{e}, GCD(r,e)=1, r<0, s_1, \dots s_\ell \in \mathbb{Z})$

$\mathcal{O}_c = \bigoplus_{n\geq 0} \mathcal{O}_c(n)$, $L \in Irr\, \mathcal{O}_c \rightsquigarrow Supp\, L$ param by $i(L), j(L) \in \mathbb{Z}_{\geq 0}$

$W\{(0, \dots 0, \underbrace{x_1 \dots x_1}_{e}, \underbrace{x_2 \dots x_2}_{e}, \dots \underbrace{x_j \dots x_j}_{e}, y_1, \dots y_i\}$.

Have seen:

$Span\{[L], i(L) \leq i\} = \bigcap \ker [E_{k_1}] \dots [E_{k_i}]$

$E_1 \dots E_e, F_1 \dots F_e$ - cat-l $\hat{\mathfrak{sl}}_e$-action on $\mathcal{O}_c$, $[\mathcal{O}_c]$ = Fock space w. multicharge $s_1 \dots s_\ell$.

Q: Rep-n th. interpr. of $j(L)$

A: via cat-l Heisenberg action - giving stand. Heisenberg action on $\mathcal{O}_c$

11.3) $L^A(e)$ - simple for $H_{r/e}(e)$ - Cherednik alg. for $\mathfrak{S}_e$

- fin. dim, has BGG resol-n $\dots \to \Delta(\Lambda^2 \mathfrak{h}) \to \Delta(\mathfrak{h}^*) \to \Delta(triv) \to L^A(e)$

- also Koszul resol-n $\rightsquigarrow$ character

More gen-l $L^A(e\mu)$, μ-part-n

• Can compute char-s using Rouquier's equiv. thm

• $L^A(e\mu) = \mu$-isot. comp. for $\mathfrak{S}_{|\mu|}$-action on $Ind_{\mathfrak{S}_e^{|\mu|}}^{\mathfrak{S}_{e|\mu|}} (L^A(e)^{\boxtimes |\mu|})$

+ using quant. Frobenius & class. Schur-Weyl duality

$\rightsquigarrow$ functors

$B_\mu : \mathcal{O}_c(n) \longrightarrow \mathcal{O}_c(n+e|\mu|)$, $Ind_{G(\ell,1,n)\times \mathfrak{S}_{e|\mu|}}^{G(\ell,1,n+e|\mu|)} (\bullet \boxtimes L^A(e\mu))$

+

derived adj-t $B_\mu^* : \mathcal{O}_c(n) \longrightarrow \mathcal{O}_c(n-e|\mu|)$

B_μ^*: $RHom_{\mathcal{O}^A_{r/e}(e|\mu|)}(L^A(e\mu), Res_{G(\ell,1,n)}^{G(\ell,1,n-e|\mu|)\times \mathfrak{S}_{e|\mu|}}(\bullet))$.

$B_1, B_2, \dots$ ⇝ Heisenberg creation op-rs on $[\mathcal{O}_c]$

$B_1^*, B_2^*, \dots$ annih-n op-rs

Span$\{[L],\ j(L)\leq j\} \subset \bigcap_{j_1+\dots+j_k > j}$ ~~Ann~~ Ker $[B_{j_1}^*]\dots[B_{j_k}^*]$

As before: have equality

<u>Ingredients:</u> • B_μ, B_μ^* commute w. E_i, F_i (transitivity of ind-n/restr-n)

• $\mathfrak{S}_k \curvearrowright B_1^{*k} L$ if $\dim L < \infty$, L-simple $\Rightarrow \mathrm{End}(B_1^k L) = \mathbb{C}\mathfrak{S}_k$

• head$(B_\mu L) \simeq$ soc$(B_\mu L)$ - simple L', mult. of this simple, $j(L') = |\mu|$ $= \dim \mu$, all other simples L'' have $j(L'') < |\mu|$ - compare w. Chuang-Rouquier thm

⇝ counting (= equality in $\subset$).

11.2) Have counting of L w. given $i(L)$ in terms of $\hat{\mathfrak{sl}}_e$-action - NOT as in Etingof's conj. ($\mathfrak{a} \subset \hat{\mathfrak{sl}}_\ell \times \mathfrak{heis}$; our case $\mathfrak{a} \simeq \hat{\mathfrak{sl}}_e \times \mathfrak{heis}$ - properly embedded!)

To pass btw diff. settings use level-rank duality

Finite dim-l setting: $\mathfrak{sl}_m \times \mathfrak{sl}_n \underset{\text{tens. prod}}{\hookrightarrow} \mathfrak{sl}_{mn} \curvearrowright \mathbb{C}^m \otimes \mathbb{C}^n$ and so $\Lambda^k(\mathbb{C}^m \otimes \mathbb{C}^n)$

Affine setting: $\mathfrak{sl}_N[t,t^{-1}] \curvearrowright \mathbb{C}^N[t,t^{-1}]$ - no centr. ext-n

~~$\mathfrak{heis} \curvearrowright \mathbb{C}^N[t,t^{-1}]$~~ basis $u_{i+Nk} = v_i \otimes t^k$

⇝ $\mathfrak{sl}_N[t,t^{-1}] \curvearrowright \Lambda^r \mathbb{C}^N[t,t^{-1}]$ - basis $u_{i_1} \wedge \dots \wedge u_{i_r}$, $i_1 > \dots > i_r$

⇝ semi-infinite limit $\Lambda^{\infty/2} \mathbb{C}^N[t,t^{-1}]$ - basis $u_{i_1} \wedge u_{i_2} \wedge \dots$

s.t. for r big enough $i_{r+1} = i_r + 1$ ⇝ well-def. op-rs $e_0, \dots, e_N, f_0, \dots, f_N$ giving

<u>Level 1</u> rep-n of $\hat{\mathfrak{sl}}_N$ - Fock space rep-n

Also $\mathbb{C}^N[t,t^{-1}]$ is $\mathfrak{heis}$-module: $b_k u_i = u_{i-Nk}$, $b_k^* u_i = u_{i+Nk}$

⇝ $\Lambda^{\infty/2} \mathbb{C}^N[t,t^{-1}]$ is irred. $\hat{\mathfrak{sl}}_N \times \mathfrak{heis}$-module $= V(\omega_0) \otimes \mathcal{F}$

Now $N = \ell e$: $\mathfrak{sl}_e[t,t^{-1}] \otimes \mathfrak{sl}_\ell[t,t^{-1}] \otimes \mathfrak{heis} \curvearrowright \mathbb{C}^{\ell e}[t,t^{-1}]$ ⇝

11.2 $\underbrace{\hat{\mathfrak{sl}}_e}_{\text{level } \ell} \times \underbrace{\hat{\mathfrak{sl}}_\ell}_{\text{level } e} \times \mathfrak{heis} \curvearrowright \Lambda^{\infty/2} \mathbb{C}^{\ell e}[t,t^{-1}]$

Weights for $\hat{\mathfrak{sl}}_\ell$: ℓ-tuple of integers $(s_1, \dots, s_\ell)$ w. $s_1 + \dots + s_\ell = e$
Wt. space ($\hat{\mathfrak{sl}}_e \times$ Heis-module) is Fock space w. mult. charge $(s_1 \dots s_\ell)$.
This level-rank duality translates Etingof's language to Shan-Vasserot language