IMPORTANT INFORMATION ON MATH 3530

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1. What this class is about

Representation theory studies *representations* of groups, associative algebras (=associative rings with a compatible vector space structure over a given field), etc., i.e. homomorphisms from a given group/algebra to the *general linear group/algebra*. For groups, this will be the group of all invertible square matrices, or, more conceptually and more generally, the group of all invertible linear operators on a vector space. For associative algebras, we consider the algebra of all matrices (or all linear operators on a vector space).

This is an introductory course. We focus on representations with complex coefficients (or with coefficients in more general fields of characteristic 0, such as the reals) and look at representations of finite groups or of finite dimensional associative algebras (mostly assumed to be *semi-simple*). The cases of groups and of associative algebras will constitute two, larger and smaller, respectively, parts of the course.

The italized terms, such as in the previous paragraph, will not be explained in this document, but will be explained in the course.

It is worth noting that there is a lot of material that can be covered in such a course and some choices need to be made. Some of the material not covered in the main part of the course will be presented as "bonuses". Here are examples of things that are **not** going to appear (at least, in a systematic way):

- Infinite groups, e.g., Lie groups and related objects such as Lie algebras.

- Connections to Physics, that are again numerous. Perhaps, most notably, so called unitary representations of groups play the role of symmetries in Quantum Mechanics. But this is again beyond the scope of this course.
- Modular Representation theory, i.e., the study of representations with coefficients in positive characteristic fields. This has been a very hot subject in the last decade but is, again, beyond the scope of this course.
- Et cetera...

The next two sections describe what I plan to cover.

2. Representations of finite groups

We will be interested in the following four, closely related questions:

- Given a finite group G, completely describe its finite dimensional representations,
- and learn how to compute their numerical invariants such as dimensions, or, even stronger, "characters".
- Applications of representations and characters to the structure theory of finite groups,
- and their connections to other parts of Mathematics.

Here is a preliminary plan for this part.

2.1. **Introduction.** I elaborate on what these bullets include.

2.2. **Basics.**

(1) Main definitions (of representations, homomorphisms, subrepresentations, etc.) and some examples.
(2) Representations of associative algebras. The group algebra.
(3) Tensor products of vector spaces and of group representations. Duals.

2.3. **Irreducible and completely reducible representations.** The irreducible representations are basic building blocks for more general representations. Completely reducible ones are those obtained from the irreducibles in the simplest possible way (for example, an irreducible representation itself is completely reducible, however paradoxical this may sound). In this part we study these classes of representations:

(1) Irreducible representations. The Schur lemma.
(2) Completely reducible representations and their basic properties and structure.
(3) Maschke's theorem: a finite dimensional representation of a finite group over the complex numbers is completely reducible.

2.4. **Characters.** Characters are functions on the group that, in a way, completely incorporate all numerical information about representations.

(1) The definition and basic properties of characters, examples.
(2) Orthogonality of characters and applications to the study of irreducible representations.

2.5. **Applications to the structure of finite groups.** Here we use characters to prove the Burnside p^aq^b theorem: that a group of this order cannot be simple, i.e., must have a nontrivial normal subgroups. This is the first (but definitely not the last) example of how the representation theory applies to the structure theory of finite groups. Along the way, we will establish the Frobenius divisibility theorem, whose proof uses similar techniques. And we digress into algebraic integers.

2.6. **The representation theory of symmetric groups.** The richest and most interesting part of the theory deals with understanding the irreducible representations of some concrete (families of) groups. The symmetric group S_n is interesting for many reasons (to be explained). In this course we classify the irreducible representations of symmetric groups and explain how to compute their characters via the Frobenius character formula. These results establish a connection between the representation theory of symmetric groups and classical objects of Algebraic Combinatorics such as partitions, Young tableaux, and symmetric polynomials. Time and energy permitting we will also discuss connections to Enumerative geometry.

3. Representations of finite dimensional semisimple associative algebras

Here is a brief outline – more details will be posted later.

In this section we will be concerned with the representation theory of finite dimensional associative algebras. We will examine how the irreducible representations look like. We will introduce the class of *semisimple* algebras – those whose representations are completely reducible – and relate them to *simple* algebras – those without nontrivial two-sided ideals. We will then study the structure of finite dimensional simple algebras over algebraically closed field. We will also explain the theorem of Frobenuis: every division algebra over $\mathbb{R}$ is either $\mathbb{R}$, or $\mathbb{C}$, or $\mathbb{H}$ – the algebra of quaternions.

4. References

Our main reference is "Introduction to Representation theory" by Etingof et al (Etingof was and is a Professor at MIT and his coauthors were high school students – or fresh out of high school – when the book was written). This an excellent book, clearly written, with many nice exercises. We are not going to cover all of it, but students are encouraged to read the remaining parts.

I expect to use two secondary references. The first is "A course in Algebra" by Vinberg – this is a textbook I used to learn Algebra when I was in college. I expect to follow some parts of that book here and there. The second book, "Young tableaux. With applications to representation theory and geometry" by Fulton is likely to be used when we talk about the representation theory of symmetric groups and their connections to Combinatorics and Geometry.

Detailed lecture notes, written in Notability, will be posted and will often contain references. The references will be posted on the class webpage as well.

5. Assignments

Three homeworks, 10% each, three in-class quizzes, 8% each, the midterm, 16%, and the final exam, 30%.

6. Prerequisites

1) A good working knowledge of Linear Algebra (which can be achieved, say, in Math 3400):

- vector spaces, subspaces, direct sums of (sub)spaces, quotient spaces on occasion;
- linear maps, their kernels, images, the rank-nullity theorems, etc.
- linear operators, eigenvalues, diagonalizability;
- Hermitian forms and Hermitian scalar products.

Knowing Multilinear Algebra is a plus, but a self-contained intro to tensor products of vector spaces will be given.

2) A good working knowledge of basics of Abstract algebra (which can be achieved, say, in Math 3500):

- Groups, subgroups, homomorphisms, normal subgroups, quotient groups, isomorphism theorems, group actions on sets, cosets.
- More on groups, such as the center, the classification of finitely generated abelian groups.
- Rings and fields with examples. Ideals in rings, homomorphisms, etc.

We will need a few things about algebraic integers, but there will be a self-contained exposition. We will also need some versions of the fundamental theorem about symmetric polynomials, but this will also be explained (perhaps, only with a sketch of proof).

3) Bonus lectures will have their own prerequisites. You could also expect to encounter side remarks in the main lectures referring to things in Analysis, Commutative algebra, etc. These are not supposed to be crucial for understanding the lectures.