IMPORTANT INFORMATION ON MATH 6030

IVAN LOSEV

Contents

1. Introduction

Representation theory studies *representations* of groups, associative algebras, Lie algebras etc., i.e. homomorphisms from a given group/algebra to the "general linear group/algebra", for groups, this will be the group of all invertible matrices, or, more conceptually and more generally, the group of all invertible linear operators on a vector space.

There are various questions about representations of a given, say, group, one can ask. The most basic question is to classify all possible representations (up to an isomorphism). This is possible in some cases, but not in general. A more narrow question, which the first one sometimes reduces to, is to classify all *irreducible* representations. If all representations (at least in a given class) are completely reducible (a.k.a. semisimple), then one can classify arbitrary representations in terms of irreducible ones. If not, then the full classification is rarely available, but one arrives at questions interesting from Category theory or Homological algebra perspectives. Yet another important question is to get the numerical information about interesting (e.g., irreducible) representations. We can ask about dimensions (for finite dimensional representations) or about finer information – suitably understood characters.

In this class we will care about representations of groups that are close to being simple and related associative and Lie algebras: the simple algebraic groups and Lie algebras themselves, finite groups of Lie type, Hecke algebras of various sorts. We will cover classical and more modern (80's and 90's) results and also mention some recent developments.

Various features of this class include:

– this is not a basic introduction to Representation theory, see the list of prerequisites below;

– I plan to post lecture notes;

– we study the representation theory both in characteristic 0 – more classical – and in positive characteristic, the subject of great current interest;

2. Topics

I plan to cover five topics:

(I) The finite dimensional representation theory of semisimple algebraic groups and their Lie algebras in zero and positive characteristic. This is the largest part. It includes:
   (1) Some basic structure theory: algebraic groups, their Lie algebras, the universal enveloping algebras, and basics on Hopf algebras.
   (2) The finite dimensional irreducible representations of the algebraic group $\mathrm{SL}_2(\mathbb{F})$ and its Lie algebra $\mathfrak{sl}_2(\mathbb{F})$, where $\mathbb{F}$ is an algebraically closed field. The case when $\mathrm{char}(\mathbb{F})=0$ is much easier, but we will also consider the much richer case when $\mathrm{char}(\mathbb{F})=p$.
   (3) It turns out that the techniques that go into studying representations of $\mathrm{SL}_2(\mathbb{F})$ and $\mathfrak{sl}_2(\mathbb{F})$ can be generalized to handle the representations of $\mathrm{SL}_n(\mathbb{F})$ and $\mathfrak{sl}_n(\mathbb{F})$ (and more general (semi)simple algebraic groups and their Lie algebras). We will discuss these generalizations.

(II) The characteristic 0 representation theory of finite groups of Lie type (our basic example is $\mathrm{GL}_n(\mathbb{F}_q)$, where $\mathbb{F}_q$ is the field with q elements) and Hecke algebras, certain associative algebras that are going to be extremely important in the subsequent topics.

(III) Topic (I) dealt with finite dimensional representations of $\mathfrak{sl}_n(\mathbb{C})$. But one can also consider certain infinite dimensional representations. The category $\mathcal{O}$ is the most important category of such representations. We will discuss definition and basic structural features of this category and also its connection to the Hecke algebras via Kazhdan-Lusztig bases.

(IV) We'll discuss "quantum groups" $U_q(\mathfrak{sl}_2)$ (certain Hopf algebras), their connection to Hecke algebras (via R-matrices) and applications of this connection to knot invariants.

(V) Above we were dealing with finite dimensional algebraic groups and their Lie algebras. There are important infinite dimensional analogs of simple Lie algebras: affine Lie algebras, such as $\hat{\mathfrak{sl}}_2$, as well as infinite dimensional analogs of Hecke algebras – affine Hecke algebras. Time permitting we will discuss these objects.

3. Prerequisites

There are two primary prerequisites and some secondary (marked with "*").

1) Linear and Multilinear algebra: linear operators and bilinear (mostly symmetric) forms, mostly over $\mathbb{C}$. Tensor products and related constructions from Multilinear algebra. This is covered in MATH 340(0) (and tensor products are also studied briefly in MATH 380(0)/500(0) in a more general setup), [V, Chapters 5,6,8], or [L, Chapters 13-16].

2) Representation theory: basic notions, constructions and results on representations of finite groups over $\mathbb{C}$ and of finite dimensional semisimple associative algebras (over an algebraically closed field). This is covered in MATH 353(0)/533(0), or [E, Sections 2-4], [L, Chapters 17,18], [V, Chapter 11] (any of these sources should be close to sufficient). Also a write-up with a reminder will be posted.

3*) Some categorical constructions studied, e.g., in MATH 380(0)/500(0). This is needed, e.g., for the category $\mathcal{O}$.

4*) Some Algebraic geometry, mostly covered in MATH 380(0)/500(0) – for the discussion of algebraic groups. We will also need some basics on affine schemes and the notion of smoothness, but this is going to be recalled in the course.

4. Grading and office hours

There will be 3-4 homeworks. As usual with my courses, they will have more points than the formal maximum. Students will need to get about 60-70 percent of the total points to score the formal maximum. There may be a final project.

There will be office hours, twice per week.

5. References

So far, in the list of references below there are three prerequisite references. As the class progresses, more references will be added.

References

[E] P. Etingof et al, *Introduction to Representation theory.* Available from the author webpage.
[H1] J. Humphreys, *Linear algebraic groups.* GTM 21, Springer.
[H2] J. Humphreys, *Introduction to Lie algebras and Representation theory.* Springer Verlag, 1972.
[L] S. Lang, *Algebra*, 3rd edition. GTM 211, Springer.
[OV] A. Onishchik, E. Vinberg, *Lie groups and algebraic groups.* Springer-Verlag, 1990.
[V] E. Vinberg, *A course in Algebra*, GSM 14, AMS.