

# IMPORTANT INFORMATION ON MATH 7720

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## 1. INTRODUCTION

The Bernstein-Gelfand-Gelfand (BGG) category  $\mathcal{O}$  is one of the most important, classical, and extensively studied categories in Representation theory. In the 90's Soergel came up with highly influential new ideas on studying this category by relating it to more elementary objects called Soergel (bi)modules that since then were applied in numerous other contexts in Representation theory and beyond. The goal of this course is to introduce category  $\mathcal{O}$ , explain Soergel's approach, as well as some generalizations including one which play an important role in the recent work of the instructor on the interface between categorifications of double affine Hecke algebras and the Langlands duality (we are not likely to cover this work, which is still mostly in progress, but we'll cover some of ideas leading to it).

## 2. SCHEDULE

TTh 1.05-2.20. The first class is Sept 3 and the last class is Dec 10. The first four classes will be on zoom.

## 3. TOPICS

Here is a preliminary plan

- (I) The first four lectures (on zoom) will be an introduction to representation theoretic, commutative algebraic and geometric aspects of the Soergel theory, this should be thought of as a detailed introduction to [S] and some subsequent work.
- (II) We will review basics on the category  $\mathcal{O}$  and its deformed decomposition, including the infinitesimal block decomposition, translation and reflection functors, and projective objects. A reference here is [H].
- (III) We will cover Soergel's results describing the category of projective objects in the principal block and some corollaries. A reference is [EMTW].
- (IV) We will review some more involved categories and equivalences that can be approached using the Soergel theory and its variants: Bezrukavnikov's equivalence, [B], Gaiitsgory's Iwahori level version of the Kazhdan-Lusztig equivalence, [G], as well as categorical versions of the double affine Hecke algebra.
- (V) We will cover some things related to what the instructor calls the Rouquier-Soergel theory. This will include highest weight categories, Rouquier-Soergel functors, sheaves on moment graphs, and possibly some other things. References are [BM] and [L].

#### 4. PREREQUISITES

Most importantly, the structure and the finite dimensional representation of semisimple Lie algebras over  $\mathbb{C}$  (“roots and weights”) up to the Weyl character formula and its proof. Next, basics of category theory, including representation theoretic aspects, as reviewed in [E, Sections 7-9] (some of this will be reviewed in class) and derived categories. Familiarity with Algebraic geometry and Commutative algebra will be useful as well.

#### 5. HOMEWORK

A few homework problems will be given and the grade will be based on homework.

#### REFERENCES

- [B] R. Bezrukavnikov, *On two geometric realizations of an affine Hecke algebra*. Publ. Math. Inst. Hautes Études Sci. 123(2016), 1–67.
- [BM] T. Braden, R. Macpherson, *From moment graphs to intersection cohomology*. Math. Ann. 321 (2001), no. 3, 533–551.
- [EMTW] B. Elias, S. Maiksumi, U. Thiel, G. Williamson, *Introduction to Soergel bimodules*. RSME Springer Ser., 5, Springer, 2020, xxv+588 pp.
- [E] P. Etingof et al, *Introduction to representation theory*. With historical interludes by Slava Gerovitch. Stud. Math. Libr., 59, AMS, Providence, RI, 2011. viii+228 pp.
- [G] D. Gaitsgory, *A conjectural extension of the Kazhdan-Lusztig equivalence*, Publ. Res. Inst. Math. Sci. 57 (2021), no. 3-4, 1227–1376.
- [H] J. Humphreys, *Representations of semisimple Lie algebras in the BGG category  $O$* . Grad. Stud. Math., 94. AMS, Providence, RI, 2008. xvi+289 pp.
- [L] I. Losev, *Quantum category  $O$  vs affine Hecke category*. arXiv:2310.03153.
- [S] W. Soergel, *Kategorie  $O$ , perverse Garben und Moduln über den Koinvarianten zur Weylgruppe*. J. Amer. Math. Soc. 3 (1990), no. 2, 421–445.