THE QUANTUM CONNECTION

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1. Review of quantum cohomology

1.1. **Genus 0 Gromov-Witten invariants.** Let X be a smooth projective variety over $\mathbb{C}$, and $\beta \in H_2(X, \mathbb{Z})$ an effective curve class. Let $\overline{\mathcal{M}}_{0,n}(X, \beta)$ denote the moduli space (stack) of genus 0 n-pointed stable maps $\mu : C \to X$ with $\mu_*([C]) = \beta$. Recall that stability means that every contracted irreducible component has at least 3 special points, where a special point is either a marked point or a point of intersection with the rest of the curve. The stack $\overline{\mathcal{M}}_{0,n}(X, \beta)$ is known to be proper by an analogue of Deligne-Mumford stable reduction for maps. Assume that X is convex, i.e. for all $\mu : \mathbb{P}^1 \to X$, the obstruction space $\mathrm{Obs}(\mu) := H^1(C, \mu^* T_X)$ vanishes. Then $\overline{\mathcal{M}}_{0,n}(X, \beta)$ is a smooth and connected stack of dimension

$$\int_\beta c_1(T_X) + \dim X + n - 3. \tag{1}$$

In particular, for X convex, $\overline{\mathcal{M}}_{0,n}(X, \beta)$ has an ordinary fundamental class $[\overline{\mathcal{M}}_{0,n}(X, \beta)]$ in this homology dimension, and one can do intersection theory. Examples of convex varieties include homogeneous varieties G/P, where G is a reductive algebraic group and P a parabolic subgroup.

Let $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,n}(X, \beta) \to X$, $i = 1, \dots, n$, be the ith evaluation morphism

$$(C, p_1, \dots, p_n, \mu) \mapsto \mu(p_i).$$

Let $\gamma_1, \dots, \gamma_n \to H^*(X, \mathbb{C})$. Then the n-point genus 0 Gromov-Witten invariant of X of class β is defined as

$$\langle \gamma_1, \dots, \gamma_n \rangle_\beta := \int_{[\overline{\mathcal{M}}_{0,n}(X,\beta)]} \mathrm{ev}_1^*(\gamma_1) \cup \dots \cup \mathrm{ev}_n^*(\gamma_n).$$

In this setting, the Gromov-Witten invariants have an enumerative interpretation. Assume that the γ_i are homogeneous classes, that $\Gamma_i \subset X$ are subvarieties in general position of class γ_i, and that $\sum \operatorname{codim} \Gamma_i = \dim \overline{\mathcal{M}}_{0,n}(X, \beta)$, so that the above integral is potentially non-zero. Then $\langle \gamma_1, \dots, \gamma_n \rangle_\beta$ is the number of n-pointed maps $\mu : \mathbb{P}^1 \to X$ of class β with $\mu(p_i) \in \Gamma_i$.

We use the notation $\langle \gamma^k \rangle_\beta$ to denote $\langle \gamma, \dots, \gamma \rangle_\beta$ where the class γ appears k times.

Example 1.1. Let [pt] be the class of a point in $\mathbb{P}^2$. Then $\langle [pt]^5 \rangle_2 = 1$ since there is a unique conic through 5 points in general position in $\mathbb{P}^2$.

We will need the divisor equation, which describes Gromov-Witten invariants for which one of the classes is a divisor class. Let $\gamma_1, \ldots, \gamma_n \in H^*(X, \mathbb{C})$ and $\gamma \in H^2(X)$. The divisor equation is

$$\langle \gamma_1, \ldots, \gamma_n, \gamma \rangle_\beta = \left(\int_\beta \gamma \right) \langle \gamma_1, \ldots, \gamma_n \rangle_\beta. \tag{2}$$

This is immediate from the enumerative interpretation above, since $\int_\beta \gamma$ is the number of choices for the image of p_{n+1}.

1.2. **Small quantum cohomology.** Let X be as in the previous section. The Kähler cone of X, denoted $K(X)$, is defined as the intersection of $H^{1,1}(X)$ with the cone in $H^2(X, \mathbb{R})$ generated by ample divisor classes in $H^2(X, \mathbb{Z})$. Define the complexified Kähler cone

$$K_\mathbb{C}(X) = \{\omega \in H^2(X, \mathbb{C}) : \mathrm{Re}(\omega) \in K(X)\}$$

and the Kähler moduli space

$$\overline{K}_\mathbb{C}(X) = K_\mathbb{C}(X)/H^2(X, 2\pi i \mathbb{Z}).$$

For simplicity, assume that $h^{2,0}(X) = 0$, so that $H^{1,1}(X) = H^2(X)$, and assume that X has no odd-dimensional cohomology. Then we have a map

$$H^2(X, \mathbb{C}) \to H^2(X, \mathbb{C})/H^2(X, 2\pi i \mathbb{Z}) \cong H^2(X, \mathbb{C}^*)$$

defined in coordinates by $t_i \mapsto q_i = e^{-t_i}$, which sends $K_\mathbb{C}(X)$ to $\overline{K}_\mathbb{C}(X)$.

Example 1.2. For $X = \mathbb{P}^n$, the ample divisor classes in $H^{1,1}(X, \mathbb{Z}) = H^2(X, \mathbb{Z}) = \mathbb{Z}$ are the positive integers, so $K_\mathbb{C}(X) = \{z : \mathrm{Re}\, z > 0\}$ and is mapped by the exponential to $\overline{K}_\mathbb{C}(X) = \mathbb{D}^*$, the punctured unit disc.

We can now define the (small) quantum product $*_\omega$ on $H^*(X.\mathbb{C})$ for each $\omega \in H^2(X, \mathbb{C})$ and study its convergence. Let $1 = T_0, T_1, \ldots, T_m \in H^*(X, \mathbb{C})$ be a homogeneous basis, and let $T_1, \ldots, T_p$ denote the classes in $H^2(X, \mathbb{C})$. Let $T^0, \ldots, T^m \in H^*(X, \mathbb{C})$ be the dual basis defined by $(T_i, T^j) = \delta_{ij}$, where $(\ ,\ )$ is the intersection pairing. Let $\omega = \sum t_i T_i$ be a class in $H^2(X, \mathbb{C})$. Then we define the quantum product with respect to ω by

$$T_i *_\omega T_j = T_i \cup T_j + \sum_{\beta > 0} \sum_k e^{-\int_\beta \omega} \langle T_i, T_j, T_k \rangle_\beta T^k.$$

Note that this expression exists as an element of $H^*(X, \mathbb{C})$ only if the infinite sum in each coefficient converges. For X Fano, each coefficient is in fact a finite sum. Indeed, in order for the Gromov-Witten $\langle T_i, T_j, T_k \rangle_\beta$ to be non-zero, one must have an equality of dimensions (see (1))

$$i + j + k = \int_\beta c_1(T_X) + \dim X;$$

since the LHS is bounded above and $c_1(T_X)$ is ample, this equality can hold for only finitely many β.

In general, the sum is infinite and may converge for $\omega \in K_{\mathbb{C}}(X)$, since then $\int_\beta \omega > 0$ and thus the coefficients $e^{-\int_\beta \omega}$ decrease as β increases. As $\omega \to +\infty$ in $K_{\mathbb{C}}(X)$, the coefficients approach 0, so convergence becomes more likely. For X Calabi-Yau, the sum is conjecturally convergent for ω sufficiently large.

We will assume X is Fano from now on, so that the quantum product is well-defined for all $\omega \in H^2(X,\mathbb{C})$. The quantum product is super-commutative by definition. As described in Barbara's lecture, linear equivalences between certain boundary divisors on $\overline{\mathcal{M}}_{0,n}(X,\beta)$ (obtained by pulling back the linear equivalences of the 3 boundary divisors on $\overline{\mathcal{M}}_{0,4}$) yield that the quantum product is associative.

2. The quantum connection

Let X be Fano. We have a well-defined quantum product $*_\omega$ for each $\omega \in H^2(X,\mathbb{C})$. Each quantum product is a generating function for the same Gromov-Witten invariants (namely the 3-point ones), but with different weights depending on ω. We would like to study all of these quantum products simultaneously. The idea to do so is very simple: let $\mathcal{H}$ be the trivial vector bundle on $H^2(X,\mathbb{C})$ with fiber $H^*(X,\mathbb{C})$. Let $\partial_i = \partial/\partial t_i \in \mathrm{Vect}(H^2(X,\mathbb{C}))$, $i = 1,\dots,p$. Then we can view T_i as a constant section of the trivial bundle $\mathcal{H}$. We will define a connection ∇ on $\mathcal{H}$ such that $\nabla_{\partial_i}(T_j) = -T_i * T_j$, where $T_i * T_j$ denotes the section of $\mathcal{H}$ whose value at the fiber over ω is $T_i *_\omega T_j$.

Definition 2.1. The *quantum connection* of X is the connection ∇ on $\mathcal{H}$ defined by

$$\nabla_{\partial_i} = \frac{\partial}{\partial t_i} - T_i * .$$

This defines a connection on the trivial bundle $\mathcal{H} = \mathcal{O}^{\oplus m+1}$ over $\mathbb{A}^p$.

Remarks 2.2.
1. By definition $\nabla_{\partial_i}(T_j) = -T_i * T_j$.
2. More invariantly, $\nabla = d - \sum_{i=1}^p A_i dt_i$, where $A_i \in \mathrm{Mat}_{m+1}(\mathbb{C})$ is the matrix whose jth column is the coefficients of $T_i * T_j$.
3. The function $e^{-\int_\beta \omega}$ on $H^2(X,\mathbb{C})$ descends to the quotient $H^2(X,\mathbb{C})/H^2(X,2\pi i\mathbb{Z}) \cong H^2(X,\mathbb{C}^*)$. Thus, so do the quantum product and the quantum connection.

Example 2.3. Let $X = \mathbb{P}^m$, and set $T_i = H^i \in H^*(X,\mathbb{C})$, where H is the hyperplane class. As was shown in Barbara's lecture,

$$\begin{aligned} T_1{*}T_0 &= T_1 \\ &\vdots \\ T_1{*}T_{m-1} &= T_m \\ T_1{*}T_m &= e^{-t}T_0. \end{aligned}$$

So ∇ is the connection on the trivial bundle $\mathcal{H} = \mathcal{O}^{\oplus m+1}$ over $\mathbb{A}^1$ defined by

$$\nabla = d - A\ dt,$$

where

$$A = \begin{pmatrix} 0 & & & & e^{-t} \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} \in \mathrm{Mat}_{m+1}(\mathbb{C}).$$

Proposition 2.4. ∇ *is flat.*

Proof. The curvature of a connection ∇ on a locally free sheaf $\mathcal{E}$ over a space X is the map

$$\nabla^2 : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}} \Omega^2_X$$

defined, for any $v, w \in \mathrm{Vect}(X)$, by

$$\nabla^2_{v,w} = \nabla_v \nabla_w - \nabla_w \nabla_v - \nabla_{[v,w]}.$$

We have $[\partial_i, \partial_j] = 0$ since partial derivatives on a vector space commute. Using the definition of ∇, for $1 \leq i, j \leq p$ we have:

$$\begin{aligned} \nabla^2_{\partial_i,\partial_j}(T_k) &= \nabla_{\partial_i}\nabla_{\partial_j}(T_k) - \nabla_{\partial_j}\nabla_{\partial_i}(T_k) \\ &= -\nabla_{\partial_i}(T_j * T_k) + \nabla_{\partial_j}(T_i * T_k) \\ &= -\frac{\partial}{\partial t_i}(T_j * T_k) + \frac{\partial}{\partial t_j}(T_i * T_k) \\ &+ T_i * (T_j * T_k) - T_j * (T_i * T_k). \end{aligned}$$

The sum of the last two terms is 0 by associativity and super-commutativity of $*$. As for the first two terms, note that by the divisor equation (2) we have:

$$T_j * T_k = T_j \cup T_k + \sum_{\beta > 0} \sum_l e^{-t_1 \int_\beta T_1} \dots e^{-t_p \int_\beta T_p} \left(\int_\beta T_j \right) \langle T_k, T_l \rangle_\beta.$$

Hence

$$\frac{\partial}{\partial t_i}(T_j * T_k) = T_j \cup T_k - \sum_{\beta > 0} \sum_l e^{-t_1 \int_\beta T_1} \dots e^{-t_p \int_\beta T_p} \left(\int_\beta T_i \right) \left(\int_\beta T_j \right) \langle T_k, T_l \rangle_\beta.$$

This expression is symmetric in i and j, so the sum of the first two terms is 0 as well. $\square$

We next view ∇ as a connection on $H^2(X, \mathbb{C}^*) \cong (\mathbb{C}^*)^p$ using Remark 2.2(3), and compute its monodromy around 0 in each factor. We will need the following general lemma (see e.g. [1], III, 1.4.1).

Lemma 2.5. *Let*

$$d + \Lambda \frac{dx}{x}$$

be a connection on $\mathcal{O}^{\oplus n}$ over a disk of radius ϵ, where $\Lambda \in \mathrm{Mat}_n(\mathcal{O})$. Suppose $Res_{x=0}(\nabla) := \Lambda(0)$ is a non-resonant matrix, i.e. no two eigenvalues differ by a non-zero integer. Then the monodromy matrix M is conjugate to $e^{-2\pi i \Lambda(0)}$.

Example 2.6. For $\lambda \in \mathbb{C}$, consider the connection

$$d + \lambda \frac{dx}{x}$$

on $\mathcal{O}$ over $\mathbb{A}^1$. This has a flat section $x^{-\lambda}$ with monodromy $e^{-2\pi i \lambda}$.

Proposition 2.7. *The monodromy of ∇ around 0 in the jth factor of $H^2(X, \mathbb{C}^*) \cong (\mathbb{C}^*)^p$ is conjugate to $e^{-2\pi i T_j \cdot}$.*

Proof. Set t_i constant for $i \neq j$. Let $q_j = e^{-t_j}$, so that $dt_j = -d\log(q_j) = -dq_j/q_j$. So we can write

$$\begin{aligned}\nabla &= d - A_j dt_j \\ &= d + A_j \frac{dq_j}{q_j}.\end{aligned}$$

The kth column of A_j is the coefficients of

$$T_j * T_k = T_j \cup T_k + \sum_{\beta > 0} e^{-t_j \int_\beta T_j} \ldots,$$

where $e^{-t_j \int_\beta T_j} = q_j^{\int_\beta T_j}$. The monodromy is independent of the choice of neighborhood of 0, so we may assume that ω lies in the punctured neighborhood $\overline{K}_{\mathbb{C}}(X)$ of 0. We may additionally normalize $T_1, \ldots, T_p$ to lie in $H^2(X, \mathbb{Z})$. Then $\int_\beta T_j$ is a positive integer since β is effective and T_j is Kähler . So $q_j^{\int_\beta T_j}$ is holomorphic in q_j, and $A_j(0) = T_j \cup \cdot$. Since $T_j \cup \cdot$ raises degree, $A_j(0)$ is nilpotent, hence non-resonant. The preceding lemma thus implies that the monodromy is conjugate to $e^{-2\pi i T_j \cdot}$. $\square$

Example 2.8. For $X = \mathbb{P}^1$, set $q = e^{-t}$. The quantum connection is

$$\begin{aligned}\nabla &= d - \begin{pmatrix} 0 & e^{-t} \\ 1 & 0 \end{pmatrix} dt \\ &= d + \begin{pmatrix} 0 & q \\ 1 & 0 \end{pmatrix} \frac{dq}{q}.\end{aligned}$$

Then

$$\mathrm{Res}_{q=0}(\nabla) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = H \cup \cdot$$

and

$$M(\nabla) \sim \begin{pmatrix} 1 & 0 \\ -2\pi i & 1 \end{pmatrix}.$$

We would now like to construct flat sections of ∇. These sections will be written in terms of a generalization of Gromov-Witten invariants known as descendant Gromov-Witten invariants, or gravitational descendants.

3. Descendant Gromov-Witten invariants

3.1. **Psi classes.**

Definition 3.1. For $1 \leq i \leq n$, let $\mathcal{L}_i$ be the line bundle on $\overline{\mathcal{M}}_{0,n}(X, \beta)$ whose fiber at $[(C, p_1, \ldots, p_n, \mu)]$ is the cotangent line $T^*_{p_i}(C)$. Let $\psi_i = c_1(\mathcal{L}_i)$.

Formally, let $\pi : \mathcal{C} \to \overline{\mathcal{M}}_{0,n}(X, \beta)$ be the universal curve with marked sections $s_1, \ldots, s_n$. Then $\mathcal{L}_i = s_i^*(T^*_\pi)$, where T^*_π is the relative cotangent sheaf of π.

For $A \sqcup B = \{1, \ldots, n\}$ and $\beta_1 + \beta_2 = \beta$, let $D(A, \beta_1 | B, \beta_2)$ denote the closure of the set of maps $(\mathbb{P}^1, p_i \in A) \cup (\mathbb{P}^1, p_j \in B) \to X$ of class β_1 on the first component and β_2 on the second. This is a boundary divisor on $\overline{\mathcal{M}}_{0,n}(X, \beta)$. For X a point, we denote the resulting boundary divisor on $\overline{\mathcal{M}}_{0,n}$ by $D(A|B)$.

Also recall that there exist forgetful morphisms

$$\overline{\mathcal{M}}_{0,n+1}(X, \beta) \to \overline{\mathcal{M}}_{0,n}(X, \beta)$$

and

$$\overline{\mathcal{M}}_{0,n}(X, \beta) \to \overline{\mathcal{M}}_{0,n}$$

defined by forgetting the last marked point (respectively, the map), then contracting unstable irreducible components.

Example 3.2.

(1) We have

$$\begin{aligned} \mathcal{M}_{0,1}(\mathbb{P}^1, 1) &\xrightarrow{\sim} \mathbb{P}^1 \\ [(C, p, \mu)] &\mapsto \mu(p). \end{aligned}$$

Under this isomorphism $\mathcal{L}_1 = T^*\mathbb{P}^1 = \mathcal{O}(-2)$, and $\psi_1 = -2H$.

(2) We have

$$\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$$

obtained by fixing p_1, p_2, p_3 to 0,1,∞ and allowing p_4 to vary. The 3 boundary components of $\overline{\mathcal{M}}_{0,4}$ arise from bubbling when p_4 collides with p_1, p_2, p_3. The cotangent line to p_1 is only affected when p_4 collides with p_1. It easily follows that $\psi_1 = D(\{1, 4\}|\{2, 3\})$.

A second method to compute ψ_1 is to consider the forgetful morphism

$$\nu : \overline{\mathcal{M}}_{0,4} \to \overline{\mathcal{M}}_{0,3} = pt$$

which forgets p_4. Examining the number of marked points on irreducible components immediately yields the following facts. If p_2 or p_3 lie on the same irreducible component C_0 as p_1, then C_0 is not contracted by ν. Otherwise, C_0 is contracted. It follows that ψ_1 is the boundary divisor separating p_1 from p_2, p_3, which again gives $\psi_1 = D(\{1, 4\}|\{2, 3\})$.

The second method used in Example 3.2(2) can be generalized to prove the following:

Lemma 3.3. *On $\overline{\mathcal{M}}_{0,n}(X,\beta)$ with $n \geq 3$,*

$$\psi_1 = \sum_{A \sqcup B = \{4,\ldots,n\}} \sum_{\beta_1+\beta_2=\beta} D(\{1\} \cup A, \beta_1 | \{2,3\} \cup B, \beta_2).$$

3.2. **Descendant invariants.**

Definition 3.4. For $\gamma_i \in H^*(X)$ and a_i non-negative integers, $i = 1,\ldots,n$, define an *n-point descendant Gromov-Witten invariant* by

$$\langle \tau_{a_1}(\gamma_1),\ldots,\tau_n(\gamma_n)\rangle_\beta := \int_{[\overline{\mathcal{M}}_{0,n}(X,\beta)]} \mathrm{ev}_1^*(\gamma_1) \cup \psi_1^{a_1} \cup \ldots \cup \mathrm{ev}_n^*(\gamma_n) \cup \psi_n^{a_n}.$$

Note that $\langle \tau_0(\gamma_1),\ldots,\tau_0(\gamma_n)\rangle_\beta = \langle \gamma_1,\ldots,\gamma_n\rangle_\beta$.

Example 3.5. By Example 3.2(1), $\langle \tau_1(1)\rangle_1^{\mathbb{P}^1} = -2$.

The divisor equation (2) can be generalized to descendant invariants as follows: for $\gamma \in H^2(X)$,

(3)

$$\langle \tau_{a+1}(\gamma_1),\gamma_2,\ldots,\gamma_n,\gamma\rangle_\beta = \left(\int_\beta \gamma\right) \langle \tau_{a+1}(\gamma_1),\gamma_2,\ldots,\gamma_n\rangle_\beta + \langle \tau_a(\gamma_1 \cup \gamma),\gamma_2,\ldots,\gamma_n\rangle_\beta.$$

We will need another recursion among the descendant invariants called a topological recursion relation. To justify it, recall the recursive structure of the boundary of $\overline{\mathcal{M}}_{0,n}(X,\beta)$: the boundary divisor $D(A,\beta_1|B,\beta_2)$ can be expressed as the image of the morphism

$$\overline{\mathcal{M}}_{0,n_1+1}(X,\beta_1) \times_X \overline{\mathcal{M}}_{0,n_2+1}(X,\beta_2) \to \overline{\mathcal{M}}_{0,n}(X,\beta),$$

where $n = n_1 + n_2$, $\beta = \beta_1 + \beta_2$, and the fiber product morphisms send a stable map to the image of the last marked point. Geometrically, this morphism glues 2 maps together along their last marked points.

Lemma 3.3 expresses ψ_1 as a sum of boundary divisors, and the above recursive structure expresses a boundary divisor as a fiber product of moduli spaces of lower dimension. Informally, integration yields the following:

$$\langle \tau_{a+1}(\gamma_1),\gamma_2,\ldots,\gamma_n\rangle_\beta = \sum_{\beta=\beta_1+\beta_2} \sum_{j=0}^{m} \langle \tau_a(\gamma_1),T_j\rangle_{\beta_1} \langle T^j,\gamma_2,\ldots,\gamma_n\rangle_{\beta_2}. \tag{4}$$

This is called a topological recursion relation for descendant invariants, since it originates from the topological recursive structure of the boundary of $\overline{\mathcal{M}}_{0,n}(X,\beta)$.

4. Flat sections

4.1. **The quantum connection ∇_z.** We will use a modification of the quantum connection on $\mathcal{H}$ defined by

$$\nabla_z := zd - \sum_{i=1}^{p} A_i dt_i,$$

where the A_i are defined in Remark 2.2(2) and z is a formal variable. This is not a connection, but is a λ-connection with $\lambda = z$ (recall that a λ-connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$ satisfies a deformed Leibniz rule $\nabla(f\sigma) = f\nabla(\sigma) + \lambda df \otimes \sigma$ for $f \in \mathcal{O}_X$ and $\sigma \in \mathcal{E}$).

Curvature of a λ-connection is defined in the same way as for a usual connection, and one can prove flatness of ∇_z in the same way as in Lemma 2.4. A section of $\mathcal{H}$ is defined to be an expression

$$\sigma = \sum_{j=0}^{m} f_j T_j$$

with $f_j \in \mathcal{O}(\mathbb{A}^p)[[z, z^{-1}]]$, and is said to be flat if $\nabla_z(\sigma) = 0$.

4.2. **Flat sections.** We will now show that certain sections of $\mathcal{H}$ are flat for ∇_z.

Proposition 4.1. *Let $\omega \in H^2(X, \mathbb{C})$. Then for $0 \leq a \leq m$, the sections of $\mathcal{H}$ defined by*

$$s_a = e^{-\omega/z} \cup T_a + \sum_{\beta > 0} e^{-\int_\beta \omega} \sum_{j=0}^{m} \sum_{n=0}^{\infty} z^{-(n+1)} \langle \tau_n(T_a \cup e^{-\omega/z}), T_j \rangle_\beta T^j$$

form a basis of flat sections for ∇_z.

Proof. (sketch; see [2], Proposition 10.2.3 for details) We want to show that

$$z \frac{\partial s_a}{\partial t_i} = T_i * s_a \tag{5}$$

for $0 \leq a \leq m$. Expand out $e^{-\int_\beta \omega}$ and $e^{-\omega/z}$, and use the divisor equation for descendant invariants (3) to rewrite s_a as

$$s_a = T_a + \sum_{\beta > 0} \sum_{j=0}^{m} \sum_{n,k=0}^{\infty} z^{-(n+1)} \frac{1}{k!} \langle \tau_n(T_a), T_j, \omega^{(k)} \rangle_\beta T^j,$$

where $\omega^{(k)}$ denotes that ω appears k times in the descendant invariant.

On the LHS of (5), writing $\omega = \sum_{i=1}^{p} t_i T_i$ and expanding the above descendant invariant as a power series in the t_i, one finds that differentiation with respect to t_i adds a T_i to the above descendant invariant. Multiplication by z changes the index τ_n to τ_{n+1}. On the RHS of (5), quantum multiplication gives an additional sum over ordinary 3-point Gromov-Witten invariants. Computing both sides of (5) and comparing the coefficients of T^j, we are reduced to showing

$$\langle \tau_{n+1}(T_a), T_j, T_i, \omega^{(k)} \rangle_\beta = \sum_{\beta_1 + \beta_2 = \beta} \sum_{k_1 + k_2 = k} \binom{k}{k_1} \langle \tau_n(T_a), T_r, \omega^{(k_1)} \rangle_{\beta_1} \langle T^r, T_j, T_i, \omega^{(k_2)} \rangle_{\beta_2}.$$

This follows easily from the 3-point topological recursion relation (4) and the divisor equation for descendants (3).

Finally, note that the degree 0 term of s_a (as a power series in the t_i) is T_a. So the s_a are linearly independent since their degree 0 terms are. $\square$

Example 4.2. As in Example 2.3, for $X = \mathbb{P}^m$ we have

$$\nabla_z = zd - \begin{pmatrix} 0 & & & & e^{-t} \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} dt.$$

Therefore, a section $\sum_{j=0}^m f_j(t, z, z^{-1}) T_j$ of $\mathcal{H}$ is flat if and only if the f_j satisfy the following system of 1st order linear PDEs (known as the quantum differential equations):

$$\begin{aligned} z\frac{df_m}{dt} &= f_{m-1} \\ &\vdots \\ z\frac{df_1}{dt} &= f_0 \\ z\frac{df_0}{dt} &= e^{-t} f_m. \end{aligned} \tag{6}$$

Define

$$\mathcal{D} = \left(z\frac{d}{dt}\right)^{m+1} - e^{-t}.$$

By the general theory of differential equations (see e.g. [1], III, 1.1.1), $\mathcal{D}f = 0$ if and only if $f_j := (zd/dt)^j f$, $j = 0, \ldots, m$, solve the system (6).

We can solve the equation $\mathcal{D}f = 0$ explicitly as follows. Define

$$S = \sum_{d \geq 0} \frac{e^{-(H/z+d)t}}{(H+z)^{m+1}(H+2z)^{m+1} \ldots (H+dz)^{m+1}},$$

and expand $S = S_m + S_{m-1}H + \ldots + S_0 H^m$ with $S_a \in \mathbb{C}[[e^{-t}, z^{-1}]]$. Then it is easy to check that $\mathcal{D}(S_a) = 0$ for $a = 0, \ldots, m$.

Consider the matrix $M \in \mathrm{Mat}_{m+1}(\mathbb{C}[[e^{-t}, z^{-1}]])$ whose ath column is given by $(zd/dt)^j S_a$, $j = 0, \ldots, m$. It is easy to check that $M = \mathrm{id}$ in degree 0. So by the above discussion, M defines a basis of solutions to the system (6).

On the other hand, by Proposition 4.1, the matrix $\Psi \in \mathrm{Mat}_{m+1}(\mathbb{C}[[e^{-t}, z^{-1}]])$ whose ath column is the components of s_a also defines a basis of solutions to the system (6). As noted at the end of the proof of Proposition 4.1, $\Psi = \mathrm{id}$ in degree 0. So by the general theory of differential equations (see e.g. [1], III, proof of 1.4.1), we conclude that $\Psi = M$.

Let us compute the last rows of these matrices explicitly in the case of $\mathbb{P}^1$. Since $H^2 = 0$, we have

$$\frac{1}{(H+kz)^2} = \frac{1}{k^2 z^2}\left(1 - \frac{2H}{kz}\right)$$

for $k = 1, \ldots, d$. Writing $S = S_1 + S_0 H$, we can calculate that

$$
\begin{aligned}
S_0 &= \sum_{d \geq 0} \left(-\frac{2}{(d!)^2} \left(1 + \frac{1}{2} + \cdots + \frac{1}{d} \right) - t \frac{1}{(d!)^2} \right) e^{-dt} z^{-(2d+1)} \\
S_1 &= \sum_{d \geq 0} \frac{1}{(d!)^2} e^{-dt} z^{-2d}.
\end{aligned}
$$

On the other side, write $s_0 = A_0 + B_0 H$ and $s_1 = A_1 + B_1 H$. By Proposition 4.1, we have

$$
\begin{aligned}
B_0 &= -tz^{-1} + \sum_{d>0} e^{-dt} \sum_{n=0}^{\infty} z^{-(n+1)} \langle \tau_n(1 - tHz^{-1}), 1 \rangle_d \\
B_1 &= 1 + \sum_{d>0} e^{-dt} \sum_{n=0}^{\infty} z^{-(n+1)} \langle \tau_n(H), 1 \rangle_d.
\end{aligned}
$$

By (1), $\dim \overline{\mathcal{M}}_{0,2}(\mathbb{P}^1, d) = 2d$. Thus, the only non-zero descendant invariants in B_0 occur for $n = 2d$ or $2d - 1$, and the only non-zero descendant invariant in B_1 occurs for $n = 2d - 1$. So the above expressions simplify to

$$
\begin{aligned}
B_0 &= -tz^{-1} + \sum_{d>0} \left(\langle \tau_{2d}(1), 1 \rangle_d - t \langle \tau_{2d-1}(H), 1 \rangle_d \right) e^{-dt} z^{-(2d+1)} \\
B_1 &= 1 + \sum_{d>0} \langle \tau_{2d-1}(H), 1 \rangle_d e^{-dt} z^{-2d}.
\end{aligned}
$$

The equality of solutions $\Psi = M$ to the quantum differential equations is

$$
\begin{pmatrix} A_0 & A_1 \\ B_0 & B_1 \end{pmatrix} = \begin{pmatrix} z \frac{d}{dt} S_0 & z \frac{d}{dt} S_1 \\ S_0 & S_1 \end{pmatrix}.
$$

Equating coefficients of powers of e^{-t} and z^{-1}, the equalities $B_0 = S_0$ and $B_1 = S_1$ yield:

$$
\begin{aligned}
\langle \tau_{2d}(1), 1 \rangle_d &= -\frac{2}{(d!)^2} \left(1 + \frac{1}{2} + \cdots + \frac{1}{d} \right) \\
\langle \tau_{2d-1}(H), 1 \rangle_d &= \frac{1}{(d!)^2}.
\end{aligned}
$$

These 2-point descendant invariants can also be computed directly by induction without any reference to the quantum connection (see [2], Example 10.1.3.1). The quantum connection organizes them as coefficients of its flat sections.

Finally, returning to the general case $X = \mathbb{P}^m$, the last rows of M and Ψ determine cohomology classes on $\mathbb{P}^m$ which are usually called the I-function and J-function of

$\mathbb{P}^m$, respectively, in the literature:

$$I_{\mathbb{P}^m} := S = \sum_{a=0}^{m} S_a H^{m-a}$$

$$J_{\mathbb{P}^m} := \sum_{a=0}^{m} \left(\int_{\mathbb{P}^m} s_a \right) H^{m-a}.$$

We have in particular shown that $I_{\mathbb{P}^m} = J_{\mathbb{P}^m}$, which is Givental's formulation of mirror symmetry for $\mathbb{P}^m$.

References

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