1 Introduction and Disclaimer

We will sketch the computation by Maulik and Okounkov of the quantum cohomology of $Hilb_n\mathbb{C}^2$.

As you will see, the proof is somewhat indirect, but the methods used apply to general quiver varieties, and yield a variety of other great results. See [3] for a more direct proof. Due to limitations in space and time, we will limit ourselves to a very brief overview and gloss over all technical points, and many statements will only be true up to a sign. We will not state results at their natural level of generality.

For the above reasons, **do not use this chapter as a technical reference!** It is only an appetizer for the main course: Quantum Groups and Quantum Cohomology, by Maulik and Okounkov [1].

2 Reminder on quantum cohomology and the Steinberg algebra

For notes, refer to the previous semester.

3 Main Result

$H^2(Hilb_n\mathbb{C}^2)$ is 1-dimensional, and generated by $c_1(\mathcal{V})$. We will sketch a proof of the following

Theorem 3.1 *Quantum multiplication by the divisor $c_1(\mathcal{V}) \in H^2(Hilb_n\mathbb{C}^2)$ is given by*

$$c_1(\mathcal{V}) \ + \ \hbar \sum_{k>0} \frac{kq^k}{1-q^k} \alpha_k \alpha_{-k} - \hbar \frac{q}{1-q} \sum_{n>0} \alpha_{-k} \alpha_k$$

The left hand summand indicates the classical cup product by $c_1(\mathcal{V})$. The rightmost term acts by a scalar on any fixed Hilbert scheme; the reader may safely ignore it for now.

Breaking from the notation of the previous talk, we will write α_k, α_{-k} for the action of the Heisenberg algebra elements $Z_{pt}[k], Z_1[-k]$ respectively. If one desires an expression purely in terms of Heisenberg operators, one can use Lehn's formula from the previous lecture to rewrite the cup product by $c_1(\mathcal{V})$.

Note that this is formally similar to the result for the Springer resolution, with the lie algebra g replaced by the Heisenberg algebra.

4 Guiding principle for the proof

As we saw last semester, the quantum cohomology of the Springer resolution is a commutative subalgebra of the non-commutative algebra generated by Steinberg correspondences and characteric classes, i.e. the graded affine Hecke algebra.

The Hilbert scheme is also a symplectic resolution, and one can similarly argue that its quantum cohomology is a commutative subalgebra of a non-commutative, non-cocommutative Hopf algebra Y, a kind of 'Yangian of the Heisenberg algebra'. Y contains the Heisenberg algebra, and acts on the Fock space representation V given by the union of all Hilbert schemes (described in the previous talk).

One may think of V as the 'basic' representation of *Y*. In fact it depends on a parameter $a \in \mathbb{C}$, and we write it as $V(a)$. Our first step is to construct the tensor products $V(a_1) \otimes ... \otimes V(a_r)$ geometrically. One can then play the tensor structure against the quantum product to determine the quantum cohomology of $Hilb_n \mathbb{C}^2$.

Though we will not use (or define) *Y* explicitly, we will occasionally refer to it as a guiding idea. We will italicize references to *Y* to underline their purely heuristic nature.

5 The moduli of framed sheaves

In this section, we construct the spaces $\mathcal{M}(r)$ whose cohomologies are the tensor products of the basic representation of Y.

We constructed $Hilb_n \mathbb{C}^2$ by symplectic reduction:

$$Hilb_n \mathbb{C}^2 = T^*(End(\mathbb{C}^n) \oplus Hom(\mathbb{C}^1, \mathbb{C}^n))//_0^\theta \ Gl(n).$$

We can similarly define

$$\mathcal{M}(r,n) = T^*(End(\mathbb{C}^n) \oplus Hom(\mathbb{C}^r, \mathbb{C}^n))//_0^\theta \ Gl(n).$$

$\mathcal{M}(r,n)$ parametrizes stable rank r framed torsion free sheaves on $\mathbb{P}^2$, with $c_2(\mathcal{F}) = n$. We will not use this interpretation, and refer readers to [2] for a proof. More importantly for our purposes, it is also a symplectic resolution.

As with the Hilbert scheme, the $\mathbb{C}^*$ action on the cotangent fibers descends to an action on $\mathcal{M}(r,n)$ dilating the symplectic form by a character $\hbar$. We will write this torus as $\mathbb{C}^*_\hbar$, and we will usually work over the field of fractions $\mathbf{k} = H^*_{\mathbb{C}^*_\hbar}(pt)_{frac}$. The action of $A = (\mathbb{C}^*)^r$ on $\mathbb{C}^r$ induces a symplectic action on $\mathcal{M}(r,n)$. Set $G = A \times \mathbb{C}^*_\hbar$.
Set

$$\mathcal{M}(r) = \bigcup_{n=0,1,2,...} \mathcal{M}(r,n).$$

Note that the same torus A acts on each component. The Heisenberg algebra acts on $H^*_G(\mathcal{M}(r))$ (in much the same way it acted on the union of hilbert schemes); we write α_k, α_{-k} for the action of the generators.

Finally, the vector space $\mathbb{C}^n$ descends to a tautological bundle $\mathcal{V}$ on $\mathcal{M}(r)$, which coincides with the usual one on the Hilbert scheme.

6 Tensor structure

In this section we construct the map intertwining the representation $H^*_G(\mathcal{M}(r))$ with $V(a_1) \otimes ... \otimes V(a_r)$.

The following is left as an exercise to the reader.

Proposition 6.1

$$\mathcal{M}(r,n)^A = \bigcup_{n_1+\ldots+n_r=n} \prod_{i=1,\ldots,r} \mathcal{M}(1,n_i)$$

We can write this more concisely as

Proposition 6.2

$$\mathcal{M}(r)^A = \mathcal{M}(1)^r$$

Using the Kunneth formula, one obtains

$$H^*_G(\mathcal{M}(r)^A) = H^*_{\mathbb{C}^*}(\mathcal{M}(1)))^{\otimes r} \otimes H^*_A(pt)$$

where the tensor product is taken over **k**. The localization theorem thus provides an isomorphism of localized cohomologies

$$H^*_G(\mathcal{M}(r))_{loc} \xrightarrow{\sim} H^*_{\mathbb{C}^*}(\mathcal{M}(1)))^{\otimes r} \otimes H^*_A(pt)_{loc}.$$

However, this is not the map we want. We will describe a different, unlocalized and degree-preserving map called the stable envelope,

$$H^*_{\mathbb{C}^*}(\mathcal{M}(1))^{\otimes r} \otimes H^*_A(pt) = H^*_G(\mathcal{M}(r)^A) \xrightarrow{Stab_C} H^*_G(\mathcal{M}(r)),$$

which depends on an ordering C of the factors to the left. *It corresponds to the intertwiner with the corresponding ordered product of the* $V(a_i)$.

6.1 Inductive definition of $Stab_+$

For simplicity we restrict our discussion to $\mathcal{M}(2)$. Since the action of A factors through its subtorus $B = \{(z,w) : zw = 1\} \subset A$, we will often implicitly use the former rather than the latter. Set $C^+ = [1,2]$ and $C^- = [2,1]$. Consider the corresponding coweights

$$\sigma_+ : \mathbb{C}^* \to B, z \to (z, z^{-1})$$
$$\sigma_- : \mathbb{C}^* \to B, z \to (z^{-1}, z)$$

Let $\gamma \in H^*_G(X^A)$ be represented by a geometric cycle $\hat{\gamma}$. Let

$$Leaf(\gamma) = \{x \in \mathcal{M}(2) : lim_{z\to 0}\sigma_+(z) \cdot x \in \hat{\gamma}.\}$$

To a first approximation, $Stab_{C^+}(\gamma) = \overline{Leaf(\gamma)}$. However, this cycle may intersect other fixed loci. The actual stable basis minimizes such intersections, in the following sense.

Given two fixed loci Z_1 and Z_2, we say $Z_1 \geq Z_2$ if $\overline{Leaf(Z_1)}$ intersects Z_2. The transitive closure of this relation defines a partial ordering of the fixed loci. Define

$$Slope(Z) = \bigcup_{Z' \leq Z} \overline{Leaf(Z')}.$$

Note that

$$\mathcal{M}(n_1, 1) \times \mathcal{M}(n - n_1, 1) = Z_1 \geq Z_2 = \mathcal{M}(n_2, 1) \times \mathcal{M}(n - n_2, 1)$$

iff $n_1 \leq n_2$.

A acts trivially on any component K of $\mathcal{M}(r)^A$, hence we have a (non-canonical) isomorphism

$$H^*_G(K) \xrightarrow{\sim} H^*_{\mathbb{C}^*}(K) \otimes H^*_A(pt).$$

Given $\gamma \in H^*_G(K)$, we define its 'A-degree'

$$deg_A(\gamma) \in \mathbb{N}$$

as the highest degree occuring in the RHS factor; it does not depend on the choice of isomorphism. let Z_1, Z_2 be components of $\mathcal{M}(r)^A$. The normal bundle to a component Z splits as a sum of dilating and contracting directions under A:

$$N_Z = N_Z^+ \oplus N_Z^-.$$

Since A is symplectic, $dim N_Z^+ = dim N_Z^- = \frac{1}{2} codim Z$. Now let $\gamma \in H^*_{\mathbb{C}^*_\hbar}(Z_1)$. Let $\iota_j : Z_j \to \mathcal{M}(r)$ be the inclusions.

Theorem 6.3 *There exists a unique $H^*_G(pt)$-linear map*

$$Stab_+ : H^*_G(\mathcal{M}(2)^A) \to H^*_G(\mathcal{M}(2))$$

satisfying the following requirements. For $\gamma \in H^*_{\mathbb{C}^*}(\mathcal{M}(2))$,

$$\iota_1^* Stab_+(\gamma)) = eu(N^+_{Z_1})\gamma$$
$$deg_A(\iota_2^* Stab_+(\gamma)) < \frac{1}{2} codim Z_2$$
$$Stab_+(\gamma) \textit{ is supported on the slope of } Z_1$$

This is achieved essentially by taking the intersection of $\gamma_2 = \overline{Leaf(\gamma)} \cap Z_2$, adding some multiple of $\overline{Leaf(\gamma_2)}$ to $\overline{Leaf(\gamma)}$, and proceeding inductively.

The above properties of $Stab_+(\gamma)$ ensure that its restriction to other fixed loci have low A-degree. We will often take a limit in the equivariant parameters for which such contributions vanish.

Example We have $\mathcal{M}(1,2) = T^*\mathbb{P}^1 \times \mathbb{C}^2$. We have $A = (\mathbb{C}^*)^2$, acting by rotations on the first factor and trivially on the second. We have

$$\mathcal{M}(1,2)^\sigma = \mathcal{M}(1,1) \times \mathcal{M}(0,1) \cup \mathcal{M}(0,1) \times \mathcal{M}(1,1) = \mathbb{C}^2 \times [0,1] \cup \mathbb{C}^2 \times [1,0].$$

We now drop the factors of $\mathbb{C}^2$; the diligent reader can insert them back in. Let $\gamma_0 \in H^*_G([0,1])$ and $\gamma_1 \in H^*_G([1,0])$ be the fundamental classes. Let L_0 be the zero-section of $T^*\mathbb{P}^1$, and let L_1 be the fiber above $[1,0]$.

Then $Stab_+(\gamma_0) = L_0 + L_1$, and $Stab_+(\gamma_1) = L_1$. Choosing $Stab_-$ reverses the roles of the two fixed points.

6.2 $Stab_+$ from an affine deformation

Here is an alternative construction of the stable basis. The reader may skip this part if he or she wishes.

First, we deform $\mathcal{M}(r)$ to the affine space

$$\mathcal{M}(r)_\lambda = T^*(End(\mathbb{C}^n) \oplus Hom(\mathbb{C}^r, C^n))//_\lambda^\theta \; Gl(n).$$

where $\lambda \in \mathbb{C}$. For $\lambda \neq 0$, this is a smooth affine space; when $r = 1$ it is the phase space of the 'rational Calogero-Moser' system. $\mathbb{C}^*_\hbar$ acts on the total space of this deformation, preserving only the fiber at $\lambda = 0$, whereas A acts fiberwise.

Let $\mathcal{M}(2)_{\mathbb{A}^1\backslash 0}$ be the total space of the deformation away from $\lambda = 0$. Consider the smooth, closed G-stable subvariety

$$L \subset \mathcal{M}(r)^A_{\mathbb{A}^1\backslash 0} \times \mathcal{M}(r)_{\mathbb{A}^1\backslash 0}$$

consisting of pairs (x, y) such that y flows to x under σ_+. Define

$$Stab_+ \subset \mathcal{M}(2)^A \times \mathcal{M}(r)$$

to be the intersection of the closure of L with the fiber at $\lambda = 0$. The resulting correspondence defines $Stab_+$.

7 R-matrix

From now on, we write $V = H^*_{\mathbb{C}^*}(\mathcal{M}(1))$ for brevity.

Define the 'R-matrix'

$$R(u) = Stab_-^{-1} \circ Stab_+.$$

Here u is the equivariant parameter of the torus B. $R(u)$ is a $\mathbb{C}(u)$-linear automorphism of $V^{\otimes 2} \otimes \mathbb{C}(u)$. Using an analogous definition for $r = 3$, one can easily check that it satisfies the spectral Yang-Baxter equation, and hence can be used to define the aforementioned 'Yangian' Y acting on the cohomology of the Hilbert scheme.

We will not pursue that route: instead we enumerate a few properties of $R(u)$. Let $Stab_+^\tau$ be the transposed correspondence, going from X to X^A.

Theorem 7.1

$$Stab_-^{-1} = Stab_+^\tau.$$

This can be proven as follows: One shows that the composition $Stab_+^\tau \circ Stab_-$ involves only proper maps, hence we may specialize equivariant parameters as we please. using the localization theorem, it is possible to express it as a composition of correspondences between fixed loci

$$\mathcal{M}(2)^A \xrightarrow{(Stab_+^\tau)^A} \mathcal{M}(2)^A \xrightarrow{(Stab_-})^A \mathcal{M}(2)^A$$

V is the direct sum of cohomologies of the components of the fixed locus. The terms of $Stab_+$ which are not block-diagonal in this decomposition have small A-degree, by the definition of the stable basis. Taking the A-equivariant parameters to infinity, all non block-diagonal contributions vanish. The diagonal contributions are easily seen to give the identity.

Using equivariant localization, one can also write $R(u)$ as a composition

$$\mathcal{M}(2)^A \xrightarrow{Stab_+^A} \mathcal{M}(2)^A \xrightarrow{(Stab_+^\tau})^A \mathcal{M}(2)^A$$

Expanding in powers of $\frac{1}{u}$, we obtain

Theorem 7.2

$$R(u) = 1 + \frac{\hbar}{u}\mathbf{r} + O\left(\frac{1}{u^2}\right)$$

where $\mathbf{r}$ *is a Steinberg operator. We call it the classical* r*-matrix.*

Using a similar properness argument, one can show

Theorem 7.3 *R(u) commutes with all Steinberg operators.*

In particular it commutes with the action of Heisenberg.

Let $Z = \mathcal{M}(0,1) \times \mathcal{M}(1) \subset \mathcal{M}(2)$. Its connected components are maximal with respect to the partial order on fixed loci. It follows that the restriction $R(u)^Z$ of the R-matrix to the cohomology of Z, i.e. $V_0 \otimes V$, has a simple form. Let N_Z be the normal bundle to Z. Let N_Z^+ and N_Z^- be the subbundles of directions contracted and dilated by σ, respectively.

Theorem 7.4

$$R(u)^Z = \frac{eu(N_Z^+ \otimes \hbar)}{eu(N_Z^+)}$$

where $\hbar$ *(abusively) denotes the trivial line bundle with weight* $\hbar$ *under the* $\mathbb{C}^*$ *action.*

One easily checks that that $N_Z^+ = \mathcal{V}$ where $\mathcal{V}$ is the tautological bundle on the second factor. Note that the euler classes involved are equivariant with respect to A, and the RHS is a series in $\frac{1}{u}$.

Theorem 7.5 $R(u)$ *is uniquely determined by its values on* $V_0 \otimes V$ *and the fact that it commutes with all Steinbergs.*

The full operator $R(u)$ is quite complicated, but we can use the above theorem to show

Theorem 7.6

$$\mathbf{r} = 1 \otimes N + N \otimes 1 + \sum_{k \neq 0} \alpha_{-k} \otimes \alpha_k. \tag{1}$$

where N *acts by multiplication by* n *on* $H^*_{\mathbb{C}^*}(\mathcal{M}(n,1))$.

8 R-matrix as a shift operator

Recall from last semester that for the Springer resolution X, one can construct 'shift operators' $S(s,q)$, for s in the coweight lattice of G, which intertwine the quantum connection of X for shifted values of the corresponding equivariant parameter:

$$S(s,q)\nabla(a) = \nabla(a+s)S(s,q).$$

Such shift operators can be defined naturally in terms of certain curve counts over an X-bundle over $\mathbb{P}^1$, following work of Seidel. One can similarly define shift operators for $\mathcal{M}(r)$, which shift the equivariant parameters of $G = \mathbb{C}^* \times A$.

One can organize these curve counts in such a way as to show

Theorem 8.1 *The operator $S(\sigma,q)$ is equal to quantum multiplication by some class $\gamma_\sigma \in QH^*_G(\mathcal{M}(r))$. It therefore commutes with quantum multiplication by any class.*

Again using a properness argument, one can show that shift operators for the symplectic torus A are expressible in terms of the R-matrix from the previous section. In particular,

Theorem 8.2

$$Stab_+^{-1} \circ S(\sigma,q) \circ Stab_+ = q^{1\otimes N} R(u) \tag{2}$$

Here $q^{1\otimes N}$ is the operator which equals the constant n on $V_k \otimes V_n$.

We can now combine the two theorems above to deduce the main result of this section. We write Q for the operator of quantum multiplication on either $\mathcal{M}(1)$ or $\mathcal{M}(2)$, sometimes with a subscript $Q_r, r = 1,2$ when helpful. Write ΔQ for the operator

$$Stab_+^{-1} \circ Q_2 \circ Stab_+$$

acting on $V \otimes V \otimes \mathbb{C}(u)$.

Theorem 8.3

$$[q^{1\otimes N} R(u), \Delta Q] = 0. \tag{3}$$

9 Computing the quantum product

We want to compute quantum multiplication by the divisor $c_1(\mathcal{V})$ in $Hilb_n\mathbb{C}^2 = \mathcal{M}(n,1)$. We proceed (roughly) by computing the coproduct of Q, and checking that Q is determined by its coproduct.

We can split Q into a classical cup product and a purely quantum part. The coproduct of the classical part may be computed by comparing our description of $R(u)^Z$ with the formula for the classical r-matrix:

$$Stab_+^{-1} \circ c_1(\mathcal{V}) \circ Stab_+ := \Delta c_1(\mathcal{V}) = c_1(\mathcal{V}) \otimes 1 + 1 \otimes c_1(\mathcal{V}) - \hbar \sum_{k>0} \alpha_k \otimes \alpha_{-k}. \tag{4}$$

ΔQ preserves the subspace

$$(V \otimes V)_n := \bigoplus_{n_1+n_2=n} V_{n_1} \otimes V_{n_2},$$

which corresponds to a connected component of $\mathcal{M}(2)$. One can further split it into a sum of components

$$\Delta_k Q : V_{n_1} \otimes V_{n_2} \to V_{n_1+k} \otimes V_{n_2-k}.$$

Using equation (4) together with a properness argument for the quantum corrections, one shows

$$\Delta_0 Q = Q \otimes 1 + 1 \otimes Q \tag{5}$$

This determines Q from ΔQ. We can determine the other components of ΔQ explicitly. Recall theorem (3):

$$[q^{1\otimes N} R(u), \Delta Q] = 0.$$

The commutator of the classical part with $R(u)$ is encoded in 4. The quantum corrections, which we write as $Q^{corrections}$, are Steinberg operators, thus commute with $R(u)$. A dash of arithmetic gives

$$\sum_k (1-q^k)\Delta_k Q^{corrections} = \hbar \sum_k (\alpha_k \otimes \alpha_{-k} - \alpha_{-k} \otimes \alpha_k)$$

This determines $\Delta_k Q$ for all k except 0. Finally, we recall the conjectured expression for Q

$$Q_{conj} = c_1(\mathcal{V}) \ + \ \hbar \sum_{k>0} \frac{kq^k}{1-q^k} \alpha_k \alpha_{-k} + \text{ scalar term } .$$

We will show it holds on $\mathcal{M}(2)$, then deduce from (5) that it holds on $\mathcal{M}(1)$. Let

$$E = Q - Q_{conj}$$

be the error term. E clearly preserves V_n. Note that E makes sense on both $\mathcal{M}(1)$ and $\mathcal{M}(2)$. Using the results above, one computes

$$\Delta E = E \otimes 1 + 1 \otimes E.$$

In particular, ΔE preserves $Z = V_0 \otimes V$. It is not hard to see that E is a Steinberg correspondence, whence

$$[R(u), \Delta E] = 0.$$

Specializing to Z, we get

$$[R(u)^Z, 1 \otimes E] = 0.$$

An operator which commutes with $R(u)^Z$ commutes with all characteristic classes of $\mathcal{V}$.

Using a resolution of the diagonal, one can show that characteristic classes generate the localized equivariant cohomology of $\mathcal{M}(r)$. It follows that E is cup product by a cohomology class. Since E has cohomological degree 0, it must be a scalar.

QED.

References

[1] D. Maulik and A. Okounkov, *Quantum Groups and Quantum Cohomology.*

[2] Hiraku Nakajima, *Lectures on hilbert schemes of points on surfaces*, no. 18, American Mathematical Soc., 1999.

[3] Andrei Okounkov and Rahul Pandharipande, *Quantum cohomology of the hilbert scheme of points in the plane*, arXiv preprint math/0411210 (2004).