Hamiltonian Reduction in Characteristic p

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1 Introduction

Today I am going to essentially repeat the constructions Ivan did in the previous section, except in characteristic p. This sounds like it adds an extra layer of difficulty. Surprisingly, the opposite is true. In particular, the "conical topology" can be replaced by the Zariski topology using the center of the algebra of differential operators. Throughout this talk we will assume, for simplicity, that we are working over the field $k = \overline{\mathbb{F}_p}$ for p much larger than the dimensions of the group G and the varieties involved, and further that both G and any varieties we use are obtained by reduction from varieties $G_{\mathbb{Z}}, X_{\mathbb{Z}}$, etc. over (a localization of) $\mathbb{Z}$. The words "smooth", "free", "reductive" and so on will be assumed to be properties of $G_{\mathbb{Q}}, X_{\overline{\mathbb{Q}}}$. There is a good sense in which many geometric statements about $X_{\overline{\mathbb{F}_p}}$ for p large are equivalent to the corresponding statements about $X_{\mathbb{Q}}$, and I will implicitly use this a few times. We will return to this more carefully at the end of the talk.

2 Poisson Center

We recall some definitions from Gufang's talk last term. Suppose Y is an affine symplectic variety. I'll use notation $k(Y)$ for functions $\Gamma(Y, \mathcal{O}_Y)$. Then $k(Y)$ is a Poisson algebra with bracket $\{,\}$. Recall the Frobenius morphism $\mathrm{Fr} : Y \to Y^{(1)}$ given on functions by $\mathrm{Fr}^*(f) = f^p$. Here $k(Y^{(1)})$ is the same ring as $k(Y)$, but with k-action twisted by the inverse frobenius automorphism on k.

Proposition 1. *The pushforward* $\mathrm{Fr}_* \mathcal{O}_Y|_y \big(= \Gamma(\mathrm{Fr}^{-1}(y), \mathcal{O}|_{\mathrm{Fr}^{-1}(Y)}\big)$ *for* $y \in Y^{(1)}$ *is a flat sheaf and the fibers* $\mathrm{Fr}_* \mathcal{O}_Y|_y$ *have Poisson brackets* $\{,\}|_y$ *such that for* $f, g \in k(Y) = \Gamma(\mathrm{Fr}_* \mathcal{O}_Y, Y^{(1)})$ *we have* $\{f, g\}|_y = \{f|_y, g|_y\}$.

Equivalently, $\mathrm{Fr}_* \mathcal{O}_Y$ is a sheaf of Poisson algebras over the scheme $Y^{(1)}$. This follows from the fact that any $\mathrm{Fr}^*(f) = f^p \in k(Y^{(1)})$ satisfies $\{f^p, g\} = p\{f^{p-1}, g\} = 0$, hence pullbacks of functions on $Y^{(1)}$ are central, and any pullback $\alpha^* \mathrm{Fr}_*(\mathcal{O}_Y)$ is canonically a Poisson algebra (in particular this is true for $\alpha : \mathrm{pt} \to Y^{(1)}$).

Suppose now $Y = T^*X$ for X affine. For the purposes of this talk, the notation DX will mean crystalline differential operators $D_{\mathrm{crys}}(X)$, i.e. over an affine X, the algebra spanned by functions and derivations, where we impose commutation relations we expect differentiations and functions to satisfy. This is a filtered quantization of the graded algebra $k(T^*(X))$. I claim DX is also canonically an algebra over a "Frobenius Center" $Z_{\mathrm{Fr}} \cong k(T^*X^{(1)})$, and this quantization is compatible with the Poisson center defined above. Namely we have the following.

Proposition 2. *Define grading on $k(T^*(X))$ by degree in the T^*-direction. There exists an injective map $\iota_Z : k(Y^{(1)}) \to DX$ whose image, called the "Frobenius Center" Z_{Fr}, lies in the center of DX, and whose associated graded is $k(Y)$ with bracket $\{,\}$. Further, quantization*

$$k(Y) \overset{q}{\dashrightarrow} DX$$

induces trivial quantization on $k(T^*(X))$, i.e. the filtration on DX by degree induces the degree filtration (scaled by p) on Z_{Fr}.

This was proven last semester: roughly, The Frobenius center is the algebra of operators that can locally near any $x \in X$ be written (for some local coordinates x_i) as $f \in k[x_i^p, \frac{\partial}{\partial x_i}^p]$. □

This means we can view DX as a bundle of algebras over $Y^{(1)}$. To point out when we are using this point of view, we will use the notation $\underline{DX}$ for this sheaf of algebras over $Y^{(1)}$. **It will be important for us that $\underline{DX}$ is flat with fibers isomorphic to matrix algebras: i.e. a *sheaf of Azumaya algebras.***

Remark 3. In the following sense $\underline{DX}$ quantizes, fiberwise, the bundle $\mathrm{Fr}_* \mathcal{O}_Y$: note that the filtration on $\underline{DX}$ induces a filtration on every fiber $\underline{DX}_y$ by $F_i\underline{DX}_y = \iota_y^* F_i DX$ where $\iota_y : \underline{DX} \to \underline{DX}_y$ is the projection, and the associated graded is canonically the fiber $\mathrm{Fr}_*(\mathcal{O}_Y)_y$

3 Frobenius for groups and universal enveloping algebras

Let G be the reduction to $\overline{\mathbb{F}_p}$ of a model over $\mathbb{Z}$ of a semisimple group with Lie algebra $\mathfrak{g}$. We recall, without proof, some new constructions we can make in characteristic p. The universal enveloping algebra, $U\mathfrak{g}$ also has a big center in characteristic p. Namely, define $\mathbb{A}(V) = \operatorname{Spec} \operatorname{Sym} V^*$ to be the affine space corresponding to V. We saw the following in Kostya's talk last term.

Theorem 4. *We have an embedding $k(\mathfrak{g}^{(1)*}) \to U\mathfrak{g}$ (sending $x \mapsto x^p - x^{[p]}$) whose image, called the p-center, $Z_{\mathfrak{g}}$ of $U\mathfrak{g}$, is central. The resulting module structure gives a flat sheaf $\underline{U\mathfrak{g}}$ of associative algebras over the affine scheme $\mathfrak{g}^{(1)*}$, whose fibers are finite-dimensional. This filtration is again compatible with quantization: namely, the PBW filtration on $U\mathfrak{g}$ induces the filtration on its associated graded $k(\mathfrak{g}^{(1)*})$ by (p times) degree of polynomial.*

Remark 5. Note that this sheaf of algebras is generally *not* Azumaya.

3.1 Frobenius Kernel

Let $G \to G^{(1)}$ be the Frobenius map. This is a map of groups, so the fiber $G_1 := \mathrm{Fr}^{-1}(e \in G^{(1)})$ is a nonreduced subgroup scheme of G, called the "Frobenius Kernel". Note that product on G_1 induces a coproduct on functions $k(G_1)$, and as $k(G_1)$ is finite-dimensional, this coproduct is equivalent to a product on the dual $k(G_1)^*$, which one should view as the group ring of the finite group scheme G_1.

This group ring is related to the p-center above, as follows. Note that the Lie algebra $T_eG_1 = T_eG$ acts on $k(G^1)$, hence also on $k(G^1)^*$ and this action is via a map of algebras $\alpha_{\mathfrak{g}} : U\mathfrak{g} \to k(G^1)^*$. This map is surjective and its kernel is $U\mathfrak{g} \cdot I_0^{(1)}$ where $I_0^{(1)} \subset k(\mathfrak{g}^{(1)*}) \subset U\mathfrak{g}$ is the ideal in the p-center corresponding to $0 \in \mathfrak{g}^{*(1)}$. (In particular, this implies that a G-action trivial on G_1 factors through $G^{(1)}$. When G_1 acts on a vector space V, the corresponding Lie algebra action of $\mathfrak{g}$ coincides with the restriction action, $\mathfrak{g} \to k(G_1)^* \to \mathrm{End}(V)$.

4 Hamiltonian reduction of differential operators.

Here we will see how the large centers from the past two sections behave under Hamiltonian reduction. Until much later, we will assume that we are always reducing with respect to the character $0 : \mathfrak{g} \to k$. This induces the character $\chi_0 : U\mathfrak{g} \to k$ taking $\mathfrak{g} \subset U\mathfrak{g}$ to 0. Define $I_0 := \ker \chi_0 \subset U\mathfrak{g}$.

Suppose X is an affine variety with G-action. Then G acts on DX and we have a moment map $\mu_{\mathrm{alg}} : U\mathfrak{g} \to DX$. We have

Proposition 6. *The p-center gets sent to the Frobenius center: $\mu_{alg}(Z_{\mathfrak{g}}) \subset Z_D$.*

This was proven in Kostya's talk. Hence we have the following diagram of algebras:

$$\begin{tikzcd} U\mathfrak{g} \arrow[r, "\mu_{\mathrm{alg}}"] & DX \\ k(\mathfrak{g}^{*(1)}) \arrow[u] \arrow[r, "*"] & k(Y^{(1)}). \arrow[u] \end{tikzcd}$$

Moreover every entry of this diagram has, compatibly, adjoint action of G: in particular, on the bottom row G acts through the quotient $G \to G^{(1)}$. The filtrations on $U\mathfrak{g}$ and DX are compatible and G-equivariant.

Finally, suppose the G-action on X is free (i.e. X is a principle G-bundle over $X//G$, equivalently any fiber has scheme-theoretically trivial stabilizer). Then we have the following theorem

Theorem 7. $R(DX, G, 0) = D(X/G)$, *in a way compatible with quantization.*

This result is true in any characteristic, and can be deduced from the classical result that $T^*X///G = T^*X/G$ where G-action on T^*X is deduced from a free G-action on X and $///$ denotes Hamiltonian reduction with respect to the zero character.

5 Hamiltonian reduction of more general quantizations.

We now abstract the setting above to a more general sheaf of algebras over a base, which might not be affine, with free G-action such that the $\mathfrak{g}$-action is given by a moment map. Namely note that the diagram of rings in the situation above gives us a diagram

$$\begin{array}{ccc} \underline{U\mathfrak{g}} & -\mu_{\mathrm{alg}}- & \underline{DX} \\ | & & | \\ \mathfrak{g}^{*(1)} & \xleftarrow{\mu_{\mathrm{geo}}} & Y^{(1)}, \end{array}$$

where $-\mu_{\mathrm{alg}}-$ stands for a map

$$\mu_{\mathrm{alg}} : {\mu_{\mathrm{geo}}}^*(\underline{U\mathfrak{g}}) \to \underline{DX},$$

(If $\underline{U\mathfrak{g}}, \underline{DX}$ were sheaves of *commutative* rings, this would be equivalent to a map of relative spectra, $\mathrm{Spec}_r(\underline{DX}) \to \mathrm{Spec}_r(\underline{U\mathfrak{g}})$).

We now formalize some of the nice properties of this picture, and say (in characteristic p) that an equivariant sheaf of algebras $\underline{A}$ over a base $Y^{(1)}$ with G-action, is equipped with a *frobenius-central moment map* when we give a map $\underline{U\mathfrak{g}} \to \Gamma(\underline{A}, Y^{(1)})$ such that on any affine Zariski open set $U \subset Y$, the restriction $\underline{U\mathfrak{g}} \to \Gamma(\underline{A}, U)$ is indeed a moment map, such that G-action on the base $Y^{(1)}$ factors through $G^{(1)}$ and such that Y is symplectic and the algebra $\underline{A}$ is a quantization of $\mathrm{Fr}_* \mathcal{O}(Y)$ with usual Poisson form, in the following sense.

Definition 8. Suppose $Y^{(1)}$ is a variety with $\mathbb{G}_m$-action factoring through $\mathbb{G}_m^{(1)}$, and $\underline{A}$ is a $\mathbb{G}_m$-equivariant sheaf of algebras on $Y^{(1)}$. We say that this $\mathbb{G}_m$-action *filters* $\underline{A}$ if it is locally finite-dimensional, weights of submodules of $\Gamma(Y^{(1)}, A)$ are bounded below, and with respect to the increasing filtration on $\Gamma(U, \underline{A})$ (induced by the grading coming from $\mathbb{G}_m$-action), the algebra $\Gamma(U, \underline{A})$ becomes graded.

Now we say $\underline{A}$ *quantizes* Y if for any affine $U \subset Y$, we have an isomorphism of graded algebras: $\mathrm{Gr}(\Gamma(U, \underline{A})) \cong \Gamma(U, \mathrm{Fr}_*(\mathcal{O}_Y))$, and this induces the trivial quantization on $\mathcal{O}_{Y^{(1)}}$, given by its grading.

6 Azumaya property of quantum Hamiltonian reduction.

In this section we prove the following theorem.

Theorem 9. *The Hamiltonian reduction* $\underline{\mathcal{A}} = R(\underline{A}, G, 0)$ *is an Azumaya algebra over* $Y^{(1)}///G^{(1)} := {\mu_{geo}}^{-1}(0)/G^{(1)}$.

The Lie algebra $\mathfrak{g}$ acts on $\underline{A}$ fiberwise over $Y^{(1)}$, and for the ideal $I_0 \subset U\mathfrak{g}$ we have that $\underline{A}/I_0\underline{A}$ is a bundle. Define ${Y^{(1)}}_0 := {\mu_{\mathrm{alg}}}^{-1}(0 \in \mathfrak{g}^{*(1)})$. Then $\underline{A}/I_0A = \underline{A} \otimes_{\mu_a^* U\mathfrak{g}} (U\mathfrak{g}/I_0)$ is supported on

$$ {\mu_{\mathrm{geo}}}^{-1}\big(\mathrm{Supp}(U\mathfrak{g}/I_0)\big) = {Y^{(1)}}_0, $$

and hence only depends on $\underline{A}|_{{Y^{(1)}}_0}$.

Now G_1 acts fiberwise, so we can form a bundle $\underline{R} := (\underline{A}/I_0)^{G_1}$, the fiberwise Hamiltonian reduction $R(\underline{A}, G_1, 0)$ over $Y^{(1)}$. Call this bundle $\underline{R}$.

Now note that

$$ R(\underline{A}, G, 0) = (\underline{A}_1/I_0\underline{A}_1)^G = \big((\underline{A}_1/I_0\underline{A}_1)^{G_1}\big)^{G^{(1)}} = R(\underline{A}, G_1, 0)^{G^{(1)}}. $$

By freeness of $G^{(1)}$-action (which follows from freeness of the associated graded action), we see that $\underline{R}$ is a pullback to a principal G-bundle of $R(\underline{A}, G, 0)$, and in particular the two have the same geometric fibers. Hence our Azumaya result is equivalent to the following lemma.

Lemma 10. *The sheaf of algebras* $R(A, G_1, 0)$ *is Azumaya over* ${\mu_{geo}}^{(-1)}(0 \in \mathbb{G}^{(1)*})$*, and has dimension* $p^{\dim(Y) - 2\dim(G)}$.

Proof. Recall the notion of Morita equivalence. Namely, two rings A, B are Morita equivalent if their categories of representations of modules are equivalent. Morita equivalence of algebras over an algebraically closed field k satisfies the following properties (their proof is an exercise).

1. An algebra A is Morita equivalent to the algebra $\mathrm{Mat}_n(A)$.
2. Any ring Morita equivalent to the base field k is a matrix algebra.

Hence to show $\underline{R}$ is Azumaya, it suffices to see that it is flat (this is formal), and to check that its fibers $\underline{R}_y$ for $y \in {\mu_{\mathrm{geo}}}^{(-1)}(0)$ are Morita equivalent to matrix algebras. This is provided by the following.

Proposition 11. *Working in the fiber over any point* $y \in mge^{(-1)}(0)$*, the module* $M_y = \underline{A}_y/I_0\underline{A}_y$*, is a free* $\underline{R}_y$*-module, and* $\mathrm{End}(\underline{R}_y) = \underline{A}_y$.

As Yi showed (and his proof works in any characteristic) we have

$$ (\underline{A}/I_0)^{\mathfrak{g}} = \mathrm{End}_{\underline{A}}(\underline{A}/I). $$

Note that the LHS is $\underline{R}$ and the RHS is $\mathrm{End}_{\underline{A}}(M)$. This means that $\underline{A}$ acts on M by $\underline{R}$-module endomorphisms, and we have a map for any fiber, a map $\rho : \underline{A}_y \to \mathrm{End}_{\underline{R}_y}(M_y)$.

We now deduce the result from the corresponding classical result. Namely, note that since G-action is free and Y is smooth, we know that the moment map is regular, i.e. $k(Y)$ is locally a free module over $k(G)$ (Yi proved this), and so ${\mu_{\mathrm{alg}}}^{(-1)}(0)$ is reduced and smooth of dimension $\dim(Y) - \dim(G)$. Further, G acts freely on ${\mu_{\mathrm{alg}}}^{-1}(0) \subset Y$, hence the geometric quotient $Y///G = \mu$ is a smooth symplectic variety of dimension $\dim(Y) - 2\dim(G)$. Because Quantum Hamiltonian reduction is compatible with quantization, we see that $\underline{R}_y$ has a filtration whose associated graded is isomorphic to the pushforward $\mathrm{Fr}_* \mathcal{O}(Y///G)_{\overline{y}}$ where $\overline{y} \in Y///G$ is the projection. This has dimension $p^{\dim(Y)-2\dim(G)}$. Thus $\rho : \underline{A}_y \to \mathrm{End}_{\underline{R}_y}(M_y)$ is a map between two unital rings of the same dimension, the first of which is simple. Thus it must be an isomorphism.

$\square$

7 What we need this for.

We now apply the general framework developed above to a specific example. Let $V = \mathfrak{gl}_n \oplus k^n$, viewed as an affine space. The group $G = GL(n)$ acts on V, T^*V. $k(T^*V)$ can be quantized to the algebra DV, and the action of G on DV is Hamiltonian, with the induced map on the Frobenius center $k(T^*V^{(1)})$ factoring through $G^{(1)}$. However, the action is not free, so the conditions from the previous section are not met; in particular, the quantum Hamiltonian reduction of DV will not be Azumaya over the geometric reduction $T^*X///G := {\mu_{\mathrm{geo}}}^{-1}(0)/G^{(1)}$.

To fix this, introduce the notation $W = T^*(V)$ and let $W^{(1)ss}$ be the set of semistable points in $W^{(1)}$ with respect to the trace character, defined in the same way as in characteristic zero. There is a moderately good GIT theory in characteristic p, but in our case since we are reducing from something defined over $\mathbb{Z}$ and p is large, all we need to know is that the semistable locus is the reduction mod p of (an integral lifting of) the semistable locus over $\mathbb{Q}$, and in particular, in this specific case, the G-action on this variety is free (in the sense used above, that $W^{(1)ss}$ is a principal G-bundle over $W^{(1)ss}//G$), as the corresponding fact is true over $\mathbb{Q}$. Now we simply take $\underline{A} = DV|_{W^{(1)ss}}$. Hence, applying theorem 9 we get an Azumaya algebra $\underline{\mathcal{A}} = R(A, G, 0)$ over $W^{(1)ss}///G = Hilb^{(1)}$.

8 Global sections and the Hilbert-Chow map.

Recall from Ivan's talk that we have $k(Hilb) = k((\mathbb{A}^2)^{(n)}) = k(\mathbb{A}^{2n})^{S_n}$, and

$$H^{\geq 1}(\mathcal{O}, Hilb) = 0. \tag{1}$$

(In Ivan's talk this was over $\mathbb{Q}$, but it implies the case over $\mathbb{F}_p$ for p large).

Equivalently, we can say that the Hilbert-Chow map $\pi_{HC} : Hilb \to \mathbb{A}^{2n}/S_n$ has $\pi_* \mathcal{O}_{Hilb} = \mathcal{O}_{\mathbb{A}^{2n}/S_n}$ and $R^{\geq 1}\pi_* \mathcal{O} = 0$. This means that $H^0(\underline{\mathcal{A}})$ will naturally be a coherent sheaf of algebras over $\mathbb{A}^{2n}/S_n$. We now observe the following.

Proposition 12. *$H^{\geq 1}(\underline{\mathcal{A}}) = 0$.*

Proof. This is a statement about sheaves, and follows from (1) by observing that $\underline{\mathcal{A}}$ is a quantization of $\mathrm{Fr}_* \, \mathcal{O}(Y///G)$ (exercise for the reader). $\square$

Further, we note that

$$\Gamma(\underline{\mathcal{A}}, Hilb) \overset{\iota}{\cong} \Gamma(\underline{\mathcal{A}}, \mu^{(-1)}_{W^{(1)ss}}(0))^G,$$

and we can use the isomorphism from the last talk, $R(D(W), G, 0) \overset{\cong}{\overline{\beta}} D(\mathbb{A}^{2n})^{S_n}$ to get a map

$$\Xi : D(\mathbb{A}^{2n})^{S_n} \to \Gamma(\underline{\mathcal{A}}, Hilb)$$

given by $\mathrm{res} \circ \iota \circ \beta^{-1}$ where $\mathrm{res} : \Gamma(\underline{D}, W^{(1)}) \to \Gamma(\underline{D}, W^{(1)ss})$ is restriction to the semistable locus. We will use this map substantially in later talks.

9 Variation: rational characters of $\mathfrak{gl}(n)$

We will in fact need a slight variation of the set-up above. Namely, we have been using everywhere the character $0 : \mathfrak{gl}(n) \to k$. We will in fact need a multiple of the trace character $\chi := \lambda \operatorname{Tr} : \mathfrak{gl}(n) \to k$. The methods of section 6 all go through, giving that $R(\underline{A}, G, \chi)$ is a sheaf of Azumaya algebras over $R(\underline{A}^{(1)}, G^{(1)}, \chi^{(1)})$, where $\chi^{(1)}$ is the point of $\mathfrak{g}^{*(1)}$ corresponding to the restriction to $Z_{\mathfrak{g}}$ of χ. This turns out to be (see [BFG]) the point $(\lambda - \lambda^p) \operatorname{Tr}$ (in general, it is the character. When this is nonzero, the classical Hamiltonian reduction turns out to be different from the Hilbert scheme (in fact, it's affine). Hence we are interested in the case where the kernel on the p-center corresponds to $0 \in \mathfrak{g}^{*(1)}$. Hence we require $\lambda - \lambda^p = 0$, i.e. $\lambda \in \mathbb{F}_p \subset \overline{\mathbb{F}_p}$, i.e. λ is "rational".

10 Addendum: Compatibility with Base Change

Here we will list briefly some facts we assumed which say that we can deduce results in characteristic p from characteristic zero. These should be sufficient to deduce characteristic-p analogues of all characteristic-zero results we needed above, where for the isomorphism $R(D(V), G, 0) \cong D(\mathbb{A}^{2n})^{S_n}$ we use the fact that both sides are finitely-generated.

Let $\mathbb{Z}' = \overline{\mathbb{Z}}[\frac{1}{(p-1)!}]$, where $\overline{\mathbb{Z}}$ is the ring of integers in $\overline{\mathbb{Q}}$ (this can be thought of as an étale neighborhood of $\mathbb{Z}$). Suppose p is a large prime, satisfying $p > c$ where c can depend on the data of a finite-type symplectic variety Y with action by a reductive group G, some algebra A and some additional finite data we will specify later, all specified over the ring $\mathbb{Z}'$. Then one can show that the following things are true.

1. For a G-equivariant Y, the categorical quotient $Y//G$ exists and $(Y//G)_k \cong Y_k/G_k$ (in particular, is noetherian). Note that for good behavior of categorical quotients in characteristic k, it suffices to assume a weaker property of G, namely geometric reductivity.

2. Any isomorphism between finitely generated algebras or finitely-generated modules between them remains an isomorphism when generators and relations are reduced mod p.

3. In particular, $A/IA_k \cong A_k/A_kI_k$.

4. $M_k^G \subset M_k^{G_k}$ in general, and this is an isomorphism when M is isomorphic (or has associated graded isomorphic to) a fixed coherent sheaf over a G-equivariant variety. (We need to assume that the variety is independent of p). Note that invariants are not always so well-behaved: for example $U\mathfrak{g}$ is a finitely-generated algebra, and its invariants depend radically on characteristic as we've seen).

5. Hence hamiltonian reduction for varieties $Y///G$ exists and $(Y///G)_k \cong (Y_k///G_k)$, and Hamiltonian reduction for algebras is independent of characteristic so long as the algebra in question quantizes a fixed coherent sheaf of rings. (In the case of DX this sheaf is $\mathcal{O}_{T^*(X)}$).

6. Fix a character χ of G. Suppose we choose a fixed lifting of X^s, X^{ss} of stable and semistable points to $\mathbb{Z}'$. Then the stable and semistable points computed over k, namely $(X_k)^s$ and $(X_k)^{ss}$ are obtained from these by base change to k.

7. In particular, a fixed G-equivariant subscheme $Y' \subset Y$ (defined over $\mathbb{Z}'$) has free G-action if and only if Y'_k has free G_k-action.

8. For a coherent sheaf E over Y and a map of sheaves $Y \to Z$, we have $Rf_*^i(E)_k \cong Rf_*^i(E_k)$ so long as f has $Rf^i(E) = 0$ for $i >> 0$, and $f_*(E_k) = f_*(E)_k$ always.

9. (Here it is important that k is algebraically closed): If G acts freely on Y and E is a G-equivariant bundle then for any point y the restriction $(E_y)_k$ is (noncanonically) isomorphic to the restriction $(E/G_{\overline{y}})_k$ where E/G is the evident quotient bundle on Y/G and $\overline{y} = y \mod G$.

   Here the bound c can depend on choice of models over $\mathbb{Z}'$ for Y, A, G, Y', E.

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