$\mathcal{O}$ & Soergel, lec 1: Overview I

1) Category $\mathcal{O}$

2) Three theorems of Soergel

1) Category $\mathcal{O}$

1.1) Definitions: $\mathbb{K}$ char. 0 field.

$\mathfrak{g}$ s/simple Lie algebra/$\mathbb{K}$ $\rightsquigarrow$ univ. enveloping $\mathcal{U}(\mathfrak{g})$

$\mathfrak{g} \supset \mathfrak{b} = \mathfrak{h} \ltimes \mathfrak{n}$ Borel

$\Lambda \subset \mathfrak{h}^*$ weight lattice.

Definition: Fix $\chi \in \mathfrak{h}^*$. Category $\mathcal{O}_\chi$ = full subcategory of $\mathcal{U}(\mathfrak{g})$-mod consisting of all M s.t.:

(i) M is fin. gen'd /$\mathcal{U}(\mathfrak{g})$

(ii) $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$ where $M_\lambda = \{m \in M \mid xm = \langle \lambda + \chi, x \rangle m \ \forall x \in \mathfrak{h}\}$

(iii) $\mathfrak{n}$ acts on M locally nilpotently: $\forall\, m \in M\ \exists\, r > 0 \mid y_1 \ldots y_r m = 0\ \forall\, y_1, \ldots y_r \in \mathfrak{n}$.

Exercise 0: $\mathcal{O}_\chi$ is an abelian subcategory of $\mathcal{U}(\mathfrak{g})$-mod (closed under (co)kernels)

Exercise 1: Modulo (i) & (ii), (iii) is equivalent to:
(iii') $\{\lambda \mid M_\lambda \neq 0\}$ is bounded from above: $\exists \lambda_1, \ldots \lambda_k \in \Lambda$ (depending on M) s.t. $M_\lambda \neq 0 \Rightarrow \exists\, i$ w. $\lambda \leq \lambda_i$ ($\overset{def}{\Longleftrightarrow}$ $\lambda_i - \lambda$ = sum of positive roots).

Exercise 2: $\dim M_\lambda < \infty \;\forall \lambda \in \Lambda$.

Examples of objects:

1) Verma modules $\Delta_\chi(\lambda)$ $(\lambda \in \Lambda)$
$\Delta_\chi(\lambda) = \mathcal{U}(\mathfrak{g}) / \mathcal{U}(\mathfrak{g}) \operatorname{Span}_{\mathbb{C}}(x - \langle \lambda + \delta, x \rangle, y : x \in \mathfrak{h}, y \in \mathfrak{n})$.

2) $\forall \lambda \in \Lambda$, $\Delta_\chi(\lambda)$ has unique irreducible quotient to be denoted $L_\chi(\lambda)$. Then $\lambda \mapsto L_\chi(\lambda) : \Lambda \to \operatorname{Irr}(\mathcal{O}_\chi)$ (the set of iso classes of irreps) is a bijection (to prove these claims is an exercise).

Exercise 3 (universal property of Verma): $\forall\, M\in \mathcal{O}_\chi$

$$\mathrm{Hom}_{\mathcal{O}_\chi}(\Delta_\chi(\lambda), M) \xrightarrow{\sim} \{m\in M_\lambda \mid \mathfrak{n}\, m=0\}$$

Remark: If $\chi'-\chi\in\Lambda$, then $\mathcal{O}_\chi \xrightarrow{\sim} \mathcal{O}_{\chi'}$, $M\mapsto M$ shifting weight decomposition. The most interesting case is $\chi=0$.

1.2) Structural features.

$W :=$ Weyl group, $\rho=\frac{1}{2}(\sum \text{positive roots})$

$Z =$ center of $\mathcal{U}(\mathfrak{g})$

Harish-Chandra: $Z \xrightarrow{\sim} \mathbb{K}[\mathfrak{h}^*]^{(W,\cdot)}: z\mapsto HC_z$, where

$w\cdot\lambda := w(\lambda+\rho)-\rho \quad (w\in W,\ \lambda\in\mathfrak{h}^*)$

so that z acts on $\Delta_\chi(\lambda)$ by $HC_z(\lambda+\chi)$.

This gives "infinitesimal block decomposition" (to be proved later)

Theorem 1: $\mathcal{O}_\chi = \bigoplus_{\xi\in \underbrace{\mathfrak{h}^*/(W,\cdot)}_{\text{space of orbits}}} \mathcal{O}_\chi^\xi$, where

$$\mathcal{O}_\chi^\xi = \{M\in\mathcal{O}_\chi \mid \forall\, m\in M\ \exists\, n>0 \mid (z-HC_z(\xi))^n m=0\ \forall z\in Z\}$$

Exercise: 1) $\Delta_\chi(\lambda), L_\chi(\lambda) \in \mathcal{O}_\chi^\xi \Leftrightarrow \xi = W\cdot(\lambda+\chi)$.

2) Use Exercise 3 in Sec 1.1 to show that $\mathcal{O}_\chi^\xi \neq 0 \Rightarrow \exists\, \lambda \in \Lambda \mid \xi = W\cdot(\lambda+\chi)$.

The following result that also will be proved later describes basic categorical properties of $\mathcal{O}_\chi^\xi$

Theorem 2: Each $\mathcal{O}_\chi^\xi$:

(i) has finitely many simple objects

(ii) $\forall\, M, N \in \mathcal{O}_\chi^\xi \Rightarrow \dim \operatorname{Hom}_{\mathcal{O}_\chi}(M,N) < \infty$

(iii) has enough projectives: $\forall\, M \in \mathcal{O}_\chi^\xi\ \exists$ projective P_M w. $P_M \twoheadrightarrow M$.

Here's an application of Thm 2. Let L be the (finite) direct sum of all simple objects in $\mathcal{O}_\chi^\xi$ & $P := P_L$ from (iii). Set $A := \operatorname{End}_{\mathcal{O}_\chi}(P)$. Then the functor $\operatorname{Hom}_{\mathcal{O}_\chi}(P,\cdot): \mathcal{O}_\chi^\xi \to \operatorname{mod}_{fd}\text{-}A$ (the category of finite dimensional right A-modules) is an equivalence). Note that $\dim A<\infty$ by ii).

Prelim. conclusion: So $\mathcal{O}^{\xi}_{\chi}$ is recovered from A or, equivalently, from the additive category $\mathcal{O}^{\xi}_{\chi}$-proj of projective objects. The Soergel theory gives an elementary ($\neq$ easy) description of $\mathcal{O}^{\xi}_{\chi}$-proj

2) Three theorems of Soergel

2.0) Digression: indecomposable objects

Let $\mathcal{C}$ be $\mathbb{K}$-linear abelian category
We say that $M\in\mathcal{C}$ is indecomposable if $M\simeq M_1\oplus M_2$ $\Rightarrow M_1$ or $M_2\simeq 0$. Assume

(*) $\forall M\in\mathcal{C} \Rightarrow \dim_{\mathbb{K}} End_{\mathcal{C}}(M)<\infty$.

One can read indecomposability of M from the structure of $End_{\mathcal{C}}(M)$. The first observation is that if $\varepsilon\in End_{\mathcal{C}}(M)$ is an idempotent, $\varepsilon\neq 0,1$, then $M\simeq \operatorname{im}\varepsilon\oplus \operatorname{im}(1-\varepsilon)$ is a nontrivial decomposition.

Exercise: Assume (*). 1) TFAE

a) M is indecomposable

b) $\mathrm{End}(M)/\mathrm{rad}\,\mathrm{End}(M)$ is a division algebra.

2) Every M decomposes as $\oplus$ of indecomposables.

Thm (Krull-Schmidt) Assume (*). Then the decomposition in 2) is unique: if $M \simeq M_1 \oplus \ldots \oplus M_\ell \simeq M_1' \oplus \ldots \oplus M_k'$, then $k=\ell$ & $\exists\, \sigma \in S_\ell$ s.t. $M_i \simeq M'_{\sigma(i)}$.

2.1) First two theorems.

For simplicity, assume $\chi = 0$, $\xi = W\cdot 0$. We write $\mathcal{O}^{pr} := \mathcal{O}_0^{W\cdot 0}$ (principal block). The simples in $\mathcal{O}^{pr}$ are $L_0(w\cdot 0)$ ($w \in W$; all these weights are distinct). For general reasons (in representation theory of fin. dim. associative algebras), $\forall w \in W\ \exists!$ <u>indecomposable</u> projective object $P_0(w\cdot 0)$ w. epimorphism to $L_0(w\cdot 0)$ ($\mathrm{Hom}_{\mathcal{O}^{pr}}(P(w\cdot 0), L(u\cdot 0))$ has $\dim = \delta_{u,w}$. Our task is to describe the algebra

$$\mathrm{End}_{\mathcal{O}^{pr}}\Big(\bigoplus_{w\in W} P(w\cdot 0)\Big).$$

The 1st theorem of Soergel is a description of the algebra $\mathrm{End}_{\mathcal{O}^{pr}}(P(w_0\cdot 0))$, where $w_0 \in W$ is the longest element sending the dominant Weyl chamber C to $-C$ so that $w_0\rho = -\rho \Rightarrow w_0\cdot\rho = -2\rho$.

In $\mathbb{K}[\mathfrak{h}^*]^W$ consider the maximal ideal $\mathbb{K}[\mathfrak{h}^*]^W_+$ of 0 (spanned by the homogeneous elements of positive degree). Define the "coinvariant algebra"

$$\mathbb{K}[\mathfrak{h}^*]^{coW} = \mathbb{K}[\mathfrak{h}^*]/\mathbb{K}[\mathfrak{h}^*]\mathbb{K}[\mathfrak{h}^*]^W_+$$

It's a graded commutative algebra of $\dim = |W|$.

Theorem 1 (Soergel): We have algebra isomorphism

$$\mathrm{End}_{\mathcal{O}^{pr}}(P_0(w_0\cdot 0)) \xrightarrow{\sim} \mathbb{K}[\mathfrak{h}^*]^{coW}$$

So we get a functor

fin. dim.

$$\mathbb{V} := \mathrm{Hom}_{\mathcal{O}^{pr}}(P(w_0\cdot 0), ?) : \mathcal{O}^{pr} \longrightarrow \mathbb{K}[\mathfrak{h}^*]^{coW}\text{-mod}_{fd}$$

It's far from being an equivalence: it sends all simples but $L_0(w_0\cdot 0)$ to 0. Nevertheless, we have

Thm 2: $\mathbb{V}$ is fully faithful on projectives:

$$\mathrm{Hom}_{\mathcal{O}^{pr}}(P_1,P_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{K}[\mathfrak{h}^*]^{coW}}(\mathbb{V}(P_1),\mathbb{V}(P_2)),\ \forall P_1,P_2\in\mathcal{O}^{pr}\text{-proj}.$$

2.2) Third theorem of Soergel.

So to describe $\mathcal{O}^{pr}$-proj it remains to determine $\mathbb{V}(P_0(w\cdot 0))$ $\forall w\in W$, an indecomposable $\mathbb{K}[\mathfrak{h}^*]^{coW}$-module (exercise: use Thm 2 + Sec 2.0).

Here are two important features of $\mathcal{O}^{pr}$-proj (to be established later):

1) Verma $\Delta_0(0)$ is projective & $\mathbb{V}(\Delta_0(0))\simeq\mathbb{K}$ (there's unique 1-dimensional $\mathbb{K}[\mathfrak{h}^*]^{coW}$-module, it's $\mathbb{K}[\mathfrak{h}^*]/(\mathfrak{h}^*)$).

2) For every simple reflection $s\in W$, there's a reflection functor $\Theta_s:\mathcal{O}^{pr}\to\mathcal{O}^{pr}$. We know that:

i) Θ_s is self-biadjoint $\Rightarrow$ exact & $\Theta_s(\mathcal{O}^{pr}\text{-proj})\subset\mathcal{O}^{pr}\text{-proj}$.

ii) If $\ell(ws)>\ell(w)$ ($\ell: W\to\mathbb{Z}_{\geqslant 0}$ is the length function), $\Rightarrow P_0(ws\cdot 0)$ appears as a direct summand in $\Theta_s P_0(w\cdot 0)$.

iii) Moreover, if $w=s_1\dots s_\ell$ is a reduced decomposition, then $P_0(w\cdot 0)$ appears with multiplicity 1 in the projective object $\Theta_{s_\ell} \dots \Theta_{s_1} \Delta_0(0)$ & all other summands are $P_0(u\cdot 0)$ w. $\ell(u) < \ell(w)$.

On the other hand, we have the endofunctor $\mathbb{K}[\mathfrak{h}^*] \otimes_{\mathbb{K}[\mathfrak{h}^*]^s} \bullet$ of $\mathbb{K}[\mathfrak{h}^*]$-mod, where $\mathbb{K}[\mathfrak{h}^*]^s = \{f \in \mathbb{K}[\mathfrak{h}^*] \mid sf = f\}$. It preserves the full subcat. $\mathbb{K}[\mathfrak{h}^*]^{coW}$-mod (b/c $\mathbb{K}[\mathfrak{h}^*]^W \subset \mathbb{K}[\mathfrak{h}^*]^s$; details are left as exercise).

Theorem 3: $\forall$ simple reflection $s \in W$, have functor isomorphism $\mathbb{V} \circ \Theta_s \xrightarrow{\sim} \mathbb{K}[\mathfrak{h}^*] \otimes_{\mathbb{K}[\mathfrak{h}^*]^s} \mathbb{V}$

This gives a description of $\mathcal{O}^{pr}$-proj as a full additive subcategory of $\mathbb{K}[\mathfrak{h}^*]^{coW}$-mod: it consists of $\oplus$'s of indecomposable summands of $\mathbb{K}[\mathfrak{h}^*]^{coW}$-modules of the form $\mathbb{K}[\mathfrak{h}^*] \otimes_{\mathbb{K}[\mathfrak{h}^*]^{s_1}} \mathbb{K}[\mathfrak{h}^*] \otimes_{\mathbb{K}[\mathfrak{h}^*]^{s_2}} \otimes \dots \otimes_{\mathbb{K}[\mathfrak{h}^*]^{s_\ell}} \mathbb{K}$. We'll revisit this in the next lecture.

The proofs of Theorems 1–3 in this section is one of main results of the course.