Q & Soergel 11: Soergel theory 3

1) $A^{W_0}\otimes_{A^W} A$ vs $Z(\mathcal{O}_A^{\Xi})$ vs $\mathrm{End}_{\mathcal{O}_A}(P_A(\lambda^-))$

1.0) Goal

Let $A := \mathbb{C}[[\mathfrak{h}^*]]$, an $S(\mathfrak{h})$-algebra via the natural embedding $\chi: S(\mathfrak{h}) \hookrightarrow \mathbb{C}[[\mathfrak{h}^*]]$ giving us category $\mathcal{O}_A$. Pick $\Xi \in \Lambda/(W,\cdot)$ & $\lambda^{\pm} \in \Xi$ as in Sec 1 of Lec 10. This gives the projective $P_A(\lambda^-) \in \mathcal{O}_A^{\Xi}$ w. $\mathrm{Hom}_{\mathcal{O}_A}(P_A(\lambda^-), L(\mu)) = \mathbb{C}^{\delta_{\lambda^-,\mu}}$ Let $W_0 = \mathrm{Stab}_W(\lambda^+ + \rho)$. Recall that $Z(\mathcal{O}_A^{\Xi})$ denotes the center of $\mathcal{O}_A^{\Xi}$. Our goal in this & next lecture is proving:

Thm: We have A-algebra isomorphisms

$$A^{W_0}\otimes_{A^W} A \xrightarrow{\sim} Z(\mathcal{O}_A^{\Xi}) \xrightarrow{\sim} \mathrm{End}_{\mathcal{O}_A}(P_A(\lambda^-))$$
$$\varphi \longmapsto \varphi_{P_A(\lambda^-)}$$

This is a stronger & more general version of Soergel's 1st thm from Sec 2 of Lec 1.

1.1) Recap

We've made some progress towards proving Thm in Lec 10. In Sec 2.3 there

(i) we produced a homomorphism $Z\otimes A\to Z(\mathcal{O}_A^{\Xi})$ (where $Z=Z(U(\mathfrak{g}))$), as well as an embedding $Z\otimes A\hookrightarrow A^{W_0}\otimes A$ whose composition w. $A^{W_0}\otimes_{A^W}A$ is surjective

(ii) We proved that the composition $\varphi_{ww_0}\colon Z\otimes A\to Z(\mathcal{O}_A^{\Xi})\to \mathrm{End}_A(\Delta_A(w\cdot\lambda^+))=A$ coincides w. the composition of the embedding $Z\otimes A\hookrightarrow A^{W_0}\otimes A$ & the homomorphism $A^{W_0}\otimes A\twoheadrightarrow A^{W_0}\otimes_{A^W}A\to A$, $a\otimes b\mapsto w(a)b$.

(iii) In Sec 2.4 for a localization B of A we considered the localization functor $\mathcal{O}_A\to\mathcal{O}_B$, $M\mapsto M_B:=B\otimes_A M$ & proved that $B\otimes_A\mathrm{Hom}_{\mathcal{O}_A}(M,M')\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{O}_B}(M_B,M'_B)$ & that $P\in\mathcal{O}_A\text{-proj}\Rightarrow P_B\in\mathcal{O}_B\text{-proj}$.

In Sec 2.5 of Lec 10 we've stated:

Proposition 1: Let $M \in \mathcal{O}_A$ have Verma filtration:

$\{0\} = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_r$ w. $M_i/M_{i-1} \simeq \Delta_A(\lambda_i)$.

For $B = \mathrm{Frac}(A)$ or $A_{\mathfrak{p}}$, where $\mathfrak{p}$ is a minimal prime ideal not of the form (z^v), the resulting filtration of M_B splits, in particular, $M_B \simeq \bigoplus_{i=1}^{r} \Delta_B(\lambda_i)$.

We'll use this to prove the following result announced in Sec 2.3 of Lec 10.

Proposition 2: The homomorphism $Z \otimes A \to Z(\mathcal{O}_A^{W\cdot\lambda})$ factors through $Z \otimes A \twoheadrightarrow A^{W_0} \otimes_{A^W} A \to Z(\mathcal{O}_A^{W\cdot\lambda})$

1.2) Proof of Proposition 1

Lemma: Let B be an arbitrary Noetherian $S(\mathfrak{h})$-algebra. Then $\mathrm{Ext}^1_{\mathcal{O}_B}(\Delta_B(\lambda), \Delta_B(\lambda)) = 0$, $\forall\, \lambda \in \Lambda$.

Proof: exercise (use that $M \mapsto M_\lambda$ is exact & $\Delta_B(\lambda)_\lambda = B$ & universal property of Vermas).

Proof of Prop 1: Thx to Lemma & the infinitesimal block decom'n for $\mathcal{O}_B$ (Sec 2.3 in Lec 5) this reduces to showing that $\forall$ equivalence class for $\sim_{\chi'}$, where χ' is the composition $S(\mathfrak{h}) \to A \to B$ is one point. This boils down to

$$\lambda_1 + \chi - w\cdot(\lambda_2+\chi) \in \mathfrak{h}^* \otimes \underbrace{\mathfrak{m}_B}_{\text{max ideal of } B} \Rightarrow w = 1.$$

We'll treat the case $B = A_{\mathfrak{p}}$ for $\mathfrak{p} \neq (\alpha^\vee)$ – the case of $B = \mathrm{Frac}(A)$ is easier.

Note that $S(\mathfrak{h}) \subset B$. Let $x_1, \dots x_n \in \mathfrak{h} \subset B$, $x^1, \dots x^n \in \mathfrak{h}^*$. Then $\chi = \sum_{i=1}^n x^i \otimes x_i \Rightarrow w\chi = \sum wx^i \otimes x_i = \sum x^i \otimes w^{-1}x_i$

$\lambda_1 + \chi - w\cdot(\lambda_2+\chi) = \lambda_1 - w\cdot\lambda_2 + \sum_{i=1}^n x^i \otimes (x_i - w^{-1}x_i)$. This element lies in $\mathfrak{h}^* \otimes \mathfrak{m}_B \Leftrightarrow$

$$x_i - w^{-1}x_i + \langle \lambda_1 - w\cdot\lambda_2, x^i\rangle \in \mathfrak{p} \quad \forall\, i.$$

Note that $\mathfrak{p} \cap S(\mathfrak{h})$ is either $\{0\}$ or a minimal prime in $S(\mathfrak{h})$. So we cannot have two linearly independent elements of $\mathbb{C} \oplus \mathfrak{h} \subset S(\mathfrak{h})$ in $\mathfrak{p} \Rightarrow$

$\dim \mathrm{Span}(x_i - w^{-1}x_i) \leq 1 \Rightarrow w = 1$ or s_α. If $w = s_\alpha$, then $\mathrm{Span}(x_i - w^{-1}x_i) = \mathbb{C}\alpha^\vee$, so $\alpha^\vee - z \in \mathfrak{p}$ for some $z \in \mathbb{C}$. But B is an A-algebra & $\alpha^\vee - z$ is invertible in A if $z \neq 0$.

This shows $\alpha^\vee \in \mathfrak{p}$ & since $\mathfrak{p}$ is minimal, $\mathfrak{m} = (\alpha^\vee)$, which we excluded. □

Important remark: If $\mathfrak{p} = (\alpha^\vee)$, then

$$\lambda_1 + \chi - w\cdot(\lambda_2 + \chi) \in \mathfrak{g}^* \otimes \mathfrak{p} \Rightarrow \begin{cases} w = 1, & \lambda_1 = \lambda_2 \\ w = s_\alpha, & \lambda_1 = s_\alpha \cdot \lambda_2 \end{cases}$$

The proof repeats that of Proposition, details are an exercise.

1.3) Proof of Proposition 2

Step 1: We show that the natural homomorphism

$$\iota: Z(O_A^\Xi) \longrightarrow A^{\oplus \Xi}, \quad z \mapsto (z_{\Delta_A(\lambda)} \in A)_{\lambda \in \Xi} \qquad (1)$$

is an embedding. There's a projective generator $\tilde{P}$ of O_A^Ξ filtered by Vermas. Let B be as in Prop 1 & η_λ, $\lambda \in \Xi$, be s.t. $\tilde{P}_B \simeq M_B$ w. $M = \bigoplus_{\lambda \in \Xi} \Delta_A(\lambda)^{\oplus n_\lambda}$. In other words, $\exists$ $\varphi: \tilde{P} \to M$, $\psi: M \to \tilde{P}$ s.t. $\varphi\psi = b\,\mathrm{id}_M$, $\psi\varphi = b\,\mathrm{id}_{\tilde{P}}$ for $b \in A$ invertible in B. We have $\varphi z_{\tilde{P}} = z_M \varphi$. So if z lies in the kernel of (1), then $z_M = 0 \Rightarrow \varphi z_{\tilde{P}} = 0 \Rightarrow b z_{\tilde{P}} = \psi\varphi z_{\tilde{P}} = 0$. On the other hand, any module in O_A filtered by Vermas is

torsion free over A, so $bt_{\tilde{P}}=0 \Rightarrow z_{\tilde{P}}=0 \Rightarrow [\tilde{P}$ is projective generator$] \Rightarrow z=0$.

Step 2: As we remarked in Sec 2.3 of Lec 10, the composition $Z\otimes A \to Z(\mathcal{O}_A^{\Xi}) \xrightarrow{z\mapsto z_{\Delta_A(\lambda)}} A$ factors through $Z\otimes A \twoheadrightarrow A^{W_0}\otimes_{A^W} A$. Taking $\oplus$ we get commutative diagram

$$\begin{array}{ccc} Z\otimes A & \longrightarrow & Z(\mathcal{O}_A^{\Xi}) \\ \downarrow & \nearrow & \downarrow \wr \\ A^{W_0}\otimes_{A^W} A & \longrightarrow & A^{\oplus \Xi} \end{array}$$

$$b\otimes a \mapsto (w(b)a)_{w\in W/W_0} \xrightarrow{\sim} \Xi$$

This yields the dashed arrow $\square$

Important remarks.

1) The bottom arrow is injective. To show this it's enough to check that the analogous homomorphism

$$\mathbb{C}[\mathfrak{h}^*]^{W_0}\otimes_{\mathbb{C}[\mathfrak{h}^*]^W}\mathbb{C}[\mathfrak{h}^*] \to \mathbb{C}[\mathfrak{h}^*]^{\oplus W/W_0} \qquad (2)$$

is injective (then we complete at 0 = tensor by A).

To prove (2) is injective note that

i) $\mathbb{C}[\mathfrak{h}^*]$ is free over $\mathbb{C}[\mathfrak{h}^*]^W \Rightarrow$ the source is torsion-free (in fact, free) over $\mathbb{C}[\mathfrak{h}^*]^W$

ii) After base change to $\mathbb{C}[\mathfrak{h}^{*,reg}]^W$, (2) is injective (in fact, an isomorphism): $\mathfrak{h}^{*,reg}/W_0 \times_{\mathfrak{h}^{*,reg}/W} \mathfrak{h}^{*,reg}$ is smooth & the map from $\coprod_{W/W_0} \mathfrak{h}^{*,reg}$ is bijective (exercise).

2) For B as in the proof we claim that $B \otimes_A C$ is iso.

Let $\tilde{E} = End_{\mathcal{O}_A}(\tilde{P})^{opp}$ so that

$$Hom_{\mathcal{O}_A}(\tilde{P}, ?): \mathcal{O}_A^{\Xi} \xrightarrow{\sim} \tilde{E}\text{-mod}_{fg}$$

& $Z(\mathcal{O}_A^{\Xi}) \xrightarrow{\sim} Z(\tilde{E})$, $z \mapsto z_{\tilde{P}}$. We have $\tilde{P}_B \xrightarrow{\sim} M_B$; by iii) in Sec 1.1,

$$B \otimes_A \tilde{E} \xrightarrow[iii)]{\sim} End_{\mathcal{O}_B}(\tilde{P}_B) \xrightarrow{\sim} End_{\mathcal{O}_B}(M_B) \xrightarrow[iii)]{\sim} B \otimes_A End_{\mathcal{O}_A}(M) \quad (3)$$

Exercise: Let E be an A-algebra that is a finitely generated & torsion-free as A-module. Let B be a localization of A. Then

(I) $B \otimes_A Z(E) \xrightarrow{\sim} Z(B \otimes_A E)$

(II) $Z(E) = E \cap Z(B \otimes_A E)$.

Note that (exercise)

$$\mathrm{Hom}_{\mathcal{O}_A}(\Delta_A(\lambda),\Delta_A(\lambda'))=0 \text{ for } \lambda\neq\lambda' \qquad (4)$$

It follows that $\mathrm{End}_{\mathcal{O}_A}(M)\xrightarrow{\sim}\bigoplus_{\lambda\in\Xi}\mathrm{Mat}_{n_\lambda}(A)$, the center is $A^{\oplus\Xi}$. One can show (important exercise) that $B\otimes_A\iota$ coincides with the restriction of (3) to the centers.

1.4) Structure of $E:=\mathrm{End}_{\mathcal{O}_A}(P_A(\lambda^-))$

Our next goal is to prove:

Proposition: $\nu: z\mapsto z_{P_A(\lambda^-)}: Z(\mathcal{O}_A^\Xi)\to E$ is iso.

We start with constructing an embedding $E\hookrightarrow A^{\oplus\Xi}$ making the following commutative

$$\begin{array}{ccc} Z(\mathcal{O}_A^\Xi) & \xrightarrow{\nu} & E \\ & \searrow^{\iota} & \downarrow \\ & & A^{\oplus\Xi} \end{array} \qquad (5)$$

Consider the object $P:=\mathcal{T}_{-\rho\to\lambda^+}\Delta_A(-\rho)$.

Lemma: $P\xrightarrow{\sim}P_A(\lambda^-)$.

Proof: Since $\mathcal{T}_{-\rho\to\lambda^+}$ is exact & admits biadjoint, P is projective, and $\mathbb{C}\otimes_A P\xrightarrow{\sim}\mathcal{T}_{-\rho\to\lambda^+}\Delta(-\rho)\xrightarrow{\sim}$ [Sec 2.2 of

Lec 9] $P(\lambda^-)$. So $\mathrm{Hom}_{\mathcal{O}_A}(P, L(\mu)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_A}(P_A(\lambda^-), L(\mu))$ $\forall \mu \in \Xi$ & $P \xrightarrow{\sim} P_A(\lambda^-)$ follows from Sec 2.3 in Lec 8 $\square$

Let $r := |\Xi|$. By Sec 1 of Lec 7 we have a filtration $\{0\} = P_0 \subsetneq P_1 \subsetneq \dots \subsetneq P_r = P$ w. $P_i/P_{i-1} \xrightarrow{\sim} \Delta_A(\lambda_i^-)$, where $\lambda_1, \dots, \lambda_r$ are all elements of Ξ w. $\lambda_i \neq \lambda_j$. (4) implies:

(a) $\forall\, \varphi \in E \simeq \mathrm{End}_{\mathcal{O}_A}(P) \Rightarrow \varphi(P_i) \subseteq P_i$ $\forall i$

(6) $\varphi \mapsto (\varphi_{P_i/P_{i-1}})_{i=1}^r \in A^{\oplus \Xi}$ is injective.

(6) is the desired embedding. Commutativity of (5) is an exercise.