O & Soergel, lecture 12: Soergel theory 4

1) $A^{W_0}\otimes_{A^W} A$ vs $Z(\mathcal{O}_A^{\Xi})$ vs $\mathrm{End}_{\mathcal{O}_A}(P_A(\lambda^-))$, cont'd

1.0) Recap

We finish proving Thm from Sec 1.0 in Lec 11.

We have a projective generator $\tilde{P}_A \in \mathcal{O}_A^{\Xi}$ first constructed in Sec 2.3 of Lec 8 as well as projective $P_A := T_{-\rho\to\lambda^+}\Delta_A(-\rho) \simeq P_A(\lambda^-)$. Set $\tilde{E}_A := \mathrm{End}_{\mathcal{O}_A}(\tilde{P}_A)^{opp}$, $E_A := \mathrm{End}_{\mathcal{O}_A}(P_A)^{opp}$. Our goal in this lecture is to prove

Thm 1: $\gamma: Z(\mathcal{O}_A^{\Xi}) \to E_A$, $z \mapsto z_{P_A}$, is iso.

Thm 2: The embedding

$$\iota: Z(\mathcal{O}_A^{\Xi}) \hookrightarrow A^{\oplus\Xi}, \quad z \mapsto (z_{\Delta_A(\lambda)})_{\lambda\in\Xi}$$

has the same image as $A^{W_0}\otimes_{A^W} A \hookrightarrow A^{\oplus\Xi}$ (from Rem 1 in Sec 1.3 of Lec 11) yielding $A^{W_0}\otimes_{A^W} A \xrightarrow{\sim} Z(\mathcal{O}_A^{\Xi})$

We'll actually describe the image (by equations)

1.1) Full faithfulness

To prove Thm 1 we need the following (cf. Prop in Sec 1 of Lec 9)

Proposition: $\mathcal{F} := \mathrm{Hom}_{\mathcal{O}_A}(P_A, ?) : \mathcal{O}_A^{\Xi} \to E_A\text{-mod}_{fg}$ is fully faithful on the projective generator $\tilde{P}_A$ of $\mathcal{O}_A^{\Xi}$.

Proof:

Note that P_A is a direct summand of $\tilde{P}_A$, let $\varepsilon \in \tilde{E}_A$ be the corresponding projector $\Rightarrow \varepsilon^2 = \varepsilon$ & $E_A = \varepsilon \tilde{E}_A \varepsilon$. Under the equivalence $\mathcal{O}_A^{\tilde{\Xi}} \xrightarrow{\sim} \tilde{E}_A\text{-mod}_{fg}$ the functor $\mathcal{F}$ becomes $M \mapsto \varepsilon M \left(\simeq \mathrm{Hom}_{\tilde{E}_A}(\tilde{E}_A \varepsilon, M) \simeq \varepsilon \tilde{E}_A \otimes_{\tilde{E}_A} M\right)$ & it admits right adjoint $\mathcal{F}^! : N \mapsto \mathrm{Hom}_{E_A}(\varepsilon \tilde{E}_A, N)$

We need to check that the adjunction unit

$$\eta_{\tilde{E}_A} : \tilde{E}_A \to \mathcal{F}^! \mathcal{F}(\tilde{E}_A) = \mathrm{End}_{E_A}(\varepsilon \tilde{E}_A)$$

is an isomorphism.

Pick a basis $x_1, \dots x_n \in \mathfrak{h}$. Set $A_k = A/(x_1, \dots x_k)$ so $A_1 = \mathbb{C}$. Set $\tilde{E}_k := A_k \otimes_A \tilde{E}_A$, etc. We'll prove $\eta_{\tilde{E}_k}$ is iso by induction.

For $k = n$, the claim follows from Proposition in Sec 1 of Lec 9 (the reduction is an exercise).

Suppose now $\eta_{\tilde{E}_{k+1}}$ is iso. Recall (Sec 2.2 of Lec 8) that $\tilde{E}_A$ is free over A. We have the following commutative diagram w. exact rows:

$$\begin{array}{ccccccc} 0 & \to & \tilde{E}_k & \xrightarrow{x_{k+1}} & \tilde{E}_k & \to & \tilde{E}_{k+1} \to 0 \\ & & \downarrow \eta_{\tilde{E}_k} & & \downarrow \eta_{\tilde{E}_k} & & \downarrow \eta_{\tilde{E}_{k+1}} \\ 0 & \to & \mathcal{F}^! \mathcal{F} \tilde{E}_k & \xrightarrow{x_{k+1}} & \mathcal{F}^! \mathcal{F} \tilde{E}_k & \to & \mathcal{F}^! \mathcal{F} E_{k+1} \end{array}$$

This forces the bottom row to be exact. Hence we get that $\eta_{\tilde{E}_k}$ is an isomorphism modulo x_{k+1}. Now note that:

1) E_k & $\varepsilon \tilde{E}_k$ are finitely generated over $A \Rightarrow$ so is $\mathcal{F}^! \mathcal{F} E_k = End_{E_k}(\varepsilon \tilde{E}_k)$. Now we use Nakayama lemma & see that $\eta_{\tilde{E}_k}$ is surjective.

2) x_{k+1} is a nonzero divisor in $\mathcal{F}^! \mathcal{F} \tilde{E}_k$. From this & $\tilde{E}_k / (x_{k+1}) \xrightarrow{\sim} \mathcal{F}^! \mathcal{F} \tilde{E}_k / (x_{k+1})$ one can deduce (exercise) that $\tilde{E}_k \xrightarrow{\sim} \mathcal{F}^! \mathcal{F} \tilde{E}_k$ which finishes the proof. □

1.1) Proof of Thm 1

In Secs 1.3 of Lec 11 we produced a commut. diagram.

$$A^{W_0}\otimes_{A^W} A \hookrightarrow Z(\mathcal{O}_A^{\Xi}) \xrightarrow{\nu} E \qquad (1)$$
$$\qquad\qquad\qquad\qquad\quad \iota \searrow \quad \downarrow$$
$$\qquad\qquad\qquad\qquad\qquad\quad A^{\oplus\Xi}$$

It's enough to construct $\zeta: E_A \to Z(\mathcal{O}_A^{\Xi})$ w. $\zeta\nu = id_{Z(\mathcal{O}_A^{\Xi})}$. Indeed, since $\iota: Z(\mathcal{O}_A^{\Xi}) \hookrightarrow A^{\oplus\Xi}$ is an iso after base change to $\mathrm{Frac}(A)$ (Rem 2 in Sec 1.3 in Lec 11) $\ker\zeta$ is A-torsion. But $E \hookrightarrow A^{\oplus\Xi}$ is torsion-free $\Rightarrow \ker\zeta = 0 \Rightarrow$ ν & ζ are mutually inverse.

To construct ζ we use $\tilde{E}_A \xrightarrow{\sim} \mathrm{End}_{E_A}(\varepsilon\tilde{E}_A) \Rightarrow$ $Z(\tilde{E}_A) \xleftarrow{\sim} Z(\mathrm{End}_{E_A}(\varepsilon\tilde{E}_A)) \leftarrow Z(E_A) : \zeta$. Note that under identification $P_A \simeq \tilde{E}_A\varepsilon \rightsquigarrow E_A \simeq \varepsilon\tilde{E}_A\varepsilon$, $\nu: z \mapsto z\varepsilon$ (exercise) The actions of z & $z\varepsilon$ on $\varepsilon\tilde{E}_A$ coincide, so $\zeta\circ\nu = id$ finishing proof of Thm 1. □

1.3) Description of the image

Set $H := \{(a_\lambda)_{\lambda\in\Xi} \in A^{\oplus\Xi} \mid a_{s_\alpha\lambda} - a_\lambda \in (\alpha^\vee)\ \forall$ positive roots $\alpha\}$

We view $Z(\mathcal{O}_A^{\Xi})$ as a subalgebra of $A^{\oplus\Xi}$ via ι.

We claim $Z(\mathcal{O}_A^{\Xi}) \subset H$

Assume $H \not\subset Z(\mathcal{O}_A^{\Xi}) \Leftrightarrow \exists z \in Z(\mathcal{O}_A^{\Xi})$, root α & $\lambda \in \Xi$ s.t.

$z_{\Delta_A(s_\alpha\cdot\lambda)} - z_{\Delta_A(\lambda)} \notin (\alpha^\vee)$. Set $A_\alpha := A/(\alpha^\vee)$, $\mathbb{K}_\alpha := \mathrm{Frac}(A_\alpha)$. So z acts by different scalars on $\Delta_{\mathbb{K}_\alpha}(\lambda)$, $\Delta_{\mathbb{K}_\alpha}(s_\alpha\cdot\lambda)$ so there are no Hom's between them. But it's classically (Verma's thesis) known that there is a nonzero homom'm. This proves $Z(\mathcal{O}_A^\Xi) \subset H$ reducing Thm 2 to:

Thm 2': $H' := A^{W_0} \otimes_{A^W} A \hookrightarrow A^{\oplus \Xi}$ coincides with H.

1.4) Reflexive modules.

To prove Thm 2' we need the notion of reflexive modules. Let A be a normal Noetherian domain & M a fin. gen'd torsion-free A-module, so that $M \hookrightarrow M_{\{0\}} := M[(A\setminus\{0\})^{-1}]$.

Definition: We say M is reflexive if $M = \bigcap_{\mathfrak{p}} M_{\mathfrak{p}}$, where the intersection is taken over the minimal prime (nonzero) ideals of A & inside $M_{\{0\}}$.

Example: free $\Rightarrow$ reflexive ("Hartogs thm").

Here is a key result about reflexive modules we need.

Lemma: Let M, N be fin. gen. A-modules, where M is reflexive & N is torsion-free. Let $\varphi \in \mathrm{Hom}_A(M,N)$. If $\varphi_{\mathfrak{p}}$ is iso $\forall$ min prime $\mathfrak{p}$, then φ is an isomorphism.

Proof: We can consider both M, N as A-submodules in $M_{\{0\}} \xrightarrow[\varphi]{\sim} N_{\{0\}}$ so that $M \subset N$. The condition that $\varphi_{\mathfrak{p}}$ is iso means $M_{\mathfrak{p}} = N_{\mathfrak{p}}$ as submodules in $N_{\{0\}}$. Since $M = \bigcap_{\mathfrak{p}} M_{\mathfrak{p}}$ & $N \subset \bigcap_{\mathfrak{p}} N_{\mathfrak{p}}$, we are done $\square$

1.5) Proof of Thm 2'

Step 1: It's enough to work over $R = \mathbb{C}[\mathfrak{h}^*]$ (& then complete at 0). We abuse the notation and use H, H' for their analogs over R, e.g. $H' = R^{W_0} \otimes_{RW} R$. The map

$$H' \longrightarrow R^{\oplus W/W_0}, \quad a \otimes b \mapsto ((wa)b)_{w \in W/W_0} \qquad (2)$$

has image in H: $(s_\alpha wa)b - (wa)b \in [f := wa](s_\alpha f - f) \subset (\alpha^\vee)$ exercise. So $H' \subset H$.

Step 2: To prove $H'=H$, it's enough to treat $W_0=\{1\}$. Indeed, W acts on $R^{\oplus W}$ via $w(r_u)_{u\in W}=(r_{uw^{-1}})_{u\in W}$ so that $R^{\oplus W/W_0}=(R^{\oplus W})^{W_0}$. The map $R\otimes_{RW}R\to R^{\oplus W}$ is W-invariant, where in the source W acts on the 1st factor (exercise). Now let $\widetilde{H}$, $\widetilde{H}'$ denote the analogs of H, H' for $W_0=\{1\}$. Then $H=\widetilde{H}^{W_0}$, $H'=\widetilde{H}'^{W_0}$ (exercise) finishing the reduction to the case $W_0=\{1\}$, which we assume from now on.

Step 3: Assume now $W=\{1,s\}$. Then $H=\{(r_1,r_s)\,|\,r_1-r_s\in(\alpha^\vee)\}=\mathrm{Span}_R\{(1,1),(\alpha^\vee,-\alpha^\vee)\}$. This is precisely $[\mathrm{im}\,R\otimes_{R^s}R\to R^{\oplus 2}]$ (exercise).

Step 4: We view H' as R-module via $r\mapsto r\otimes 1$.

As we've mentioned in Rem 1) in Sec 1.3, Lec 11 H' is free over R hence reflexive. Thx to Lemma in Sec. 1.4 applied to the inclusion $H'\hookrightarrow H$ it's enough to show that $H'_\mathfrak{p}=H_\mathfrak{p}$ $\forall$ min. primes $\mathfrak{p}$. Moreover, we've seen that after base change to $\mathbb{C}[\mathfrak{h}^{*,\mathrm{reg}}]^W$, $H'\hookrightarrow R^{\oplus W}$ becomes iso. So, it's enough to consider $\mathfrak{p}=(\alpha^\vee)$ for a root α.

$\neq$

Step 5: Thx to $R\otimes_{R^W} R \hookrightarrow R^{\oplus W}$, the source is reduced $\Rightarrow$ $\mathfrak{h}^*\times_{\mathfrak{h}^*/W}\mathfrak{h}^* = \bigcup_{w\in W}\Gamma_w \subset \mathfrak{h}^*\times\mathfrak{h}^*$, where $\Gamma_w := \{(wx,x)\,|\,x\in\mathfrak{h}^*\}$, & $R\otimes_{R^W} R \hookrightarrow R^{\oplus W}$ is the pullback to $\bigsqcup_{w\in W}\Gamma_w$. So $\forall$ root β we factorize

$$\bigsqcup_{w\in W}\Gamma_w \to \bigsqcup_{x\in\langle s_\beta\rangle\backslash W}\underbrace{\left(\bigcup_{w\in x}\Gamma_w\right)}_{\text{in } \mathfrak{h}^*\times\mathfrak{h}^*} \to \bigcup_{w\in W}\Gamma_w \qquad (3)$$

Two comments are in order:

1) The 2nd arrow is an isomorphism over $\mathfrak{h}^*\setminus\bigcup_{\alpha\neq\beta}\ker\alpha^\vee$ (this removes all pairwise intersections except over $\ker\beta^\vee$)

2) $\mathbb{C}\left[\bigsqcup_{x\in\langle s_\alpha\rangle\backslash W}\left(\bigcup_{w\in x}\Gamma_w\right)\right] = \bigoplus_{x\in\langle s_\alpha\rangle\backslash W}\mathbb{C}\left[\bigcup_{w\in x}\Gamma_w\right] = [\text{Step 3}]$
$= \{(r_w)\in R^{\oplus W}\,|\,r_{s_\beta w} - r_w \in (\beta^\vee)\ \forall w\} =: H^\beta$

Step 6: We can finish the argument. Note that $H = \bigcap_\alpha H^\alpha \Rightarrow H_{(\beta^\vee)} = \bigcap_\alpha H^\alpha_{(\beta^\vee)} = [H^\alpha_{(\beta^\vee)} = R^{\oplus W}_{(\beta^\vee)} \text{ if } \beta\neq\alpha]$ $= H^\beta_{(\beta^\vee)}$. On the other hand, $\prod_{\alpha\neq\beta}\alpha^\vee \notin (\beta^\vee)$ so 1) in Step 5 shows that $H'_{(\beta^\vee)} = H^\beta_{(\beta^\vee)}$. This verifies the claim in Step 3 & finishes the proof. □