O & Soergel, lecture 13: Soergel theory 5

1) Soergel's 3rd theorem

2) Applications to Soergel bimodules

1.0) Recap

Let $A = \mathbb{C}[[\mathfrak{h}^*]] \xleftarrow{\chi} S(\mathfrak{h}) \xrightarrow{\sim} \mathcal{O}_A$, $\Xi \subset \Lambda/(W,\cdot)$, $\lambda^\pm \in \Xi$, $P_A(\lambda^-) \simeq \mathcal{T}_{-\rho\to\lambda^+}\Delta_A(-\rho)$, $W_\circ = Stab_W(\lambda^+ + \rho)$ be as in Sec 1.0 of Lec 11. In Lecs 11 & 12 we've proved

Thm: We have A-algebra isomorphisms

$$A^{W_\circ} \otimes_{A^W} A \xrightarrow{\sim} Z(\mathcal{O}_A^\Xi) \xrightarrow{\sim} End_{\mathcal{O}_A}(P_A(\lambda^-))$$

$$\varphi \longmapsto \varphi_{P_A(\lambda^-)}$$

In Sec 2.3 of Lec 10 we've introduced:

$$\hat{\mathbb{V}}_\lambda := Hom_{\mathcal{O}_A}(\Delta_A(-\rho), \mathcal{T}_{\lambda^+\to -\rho}(?)) = Hom_{\mathcal{O}_A}(\mathcal{T}_{-\rho\to\lambda^+}\Delta_A(-\rho), ?)$$

$$\mathcal{O}_A^\Xi \longrightarrow Z(\mathcal{O}_A^\Xi)\text{-}mod_{fg},$$

where the action of $Z(\mathcal{O}_A^\Xi)$ on the target is induced from the natural action on $\mathcal{T}_{-\rho\to\lambda^+}\Delta_A(-\rho) = P_A(\lambda^-)$, which by Thm is the natural action of $End_A(P_A(\lambda^-))^{opp}$.

Since $P_A(\lambda^-)$ is projective & flat over A (exercise) we get

Corollary: $\mathrm{End}_{\mathcal{O}_0}(P_0(\lambda^-)) \simeq \mathbb{C} \otimes_A \mathrm{End}_{\mathcal{O}_A}(P_A(\lambda^-)) \simeq A^{W_0} \otimes_{A^W} \mathbb{C}$

This implies Soergel's thm from Sec 2.1 in Lec 1.

1.1) Goal

What remains from Sec 2 of Lec 1 is Soergel's 3rd Thm: for dominant λ^+, $\mathbb{V}_\lambda \circ \textcircled{H}_i \simeq \mathbb{C}[\mathfrak{h}^*] \otimes_{\mathbb{C}[\mathfrak{h}^*]^S} \mathbb{V}_\lambda$.

Recall that $\textcircled{H}_i = T_{\lambda^i \to \lambda^+} \circ T_{\lambda^+ \to \lambda^i}$, where $\langle \lambda^i, \alpha_j^\vee \rangle := -\delta_{ij}$

So more generally, we should understand the compatibility between translation functors & Soergel functors.

Let $\lambda_1, \lambda_2 \in \Lambda$ be s.t. $\lambda_2 + \rho$, $\lambda_1 - \lambda_2$ are dominant $\rightsquigarrow$ biadj't translation functors $T_{\lambda_1 \to \lambda_2} : \mathcal{O}_A^{\Xi_1} \rightleftarrows \mathcal{O}_A^{\Xi_2} : T_{\lambda_2 \to \lambda_1}$, where $\Xi_i := W \cdot \lambda_i$. Set $W_i = \mathrm{Stab}_W(\lambda_i + \rho)$ so that $W_1 \subset W_2 \Rightarrow$ $A^{W_2} \subset A^{W_1} \rightsquigarrow Z^{\Xi_1} \hookrightarrow Z^{\Xi_2}$ ($Z^{\Xi_i} := Z(\mathcal{O}_A^{\Xi_i})$). Consider the deformed Soergel functors

$$\hat{\mathbb{V}}_{\lambda_i} : \mathcal{O}_A^{\Xi_i} \longrightarrow Z^{\Xi_i}\text{-mod}_{fg}, \quad M \mapsto \mathrm{Hom}_{\mathcal{O}_A}(P_A(\lambda_i^-), M)$$

The 3rd theorem of Soergel follows (exercise) from

Thm: The following two diagrams are commutative

1)
$$\begin{array}{ccc} \mathcal{O}_A^{\Xi_1} & \xrightarrow{T_{\lambda_1\to\lambda_2}} & \mathcal{O}_A^{\Xi_2} \\ \downarrow \hat{\mathbb{V}}_{\lambda_1} & & \downarrow \hat{\mathbb{V}}_{\lambda_2} \\ Z^{\Xi_1}\text{-}\mathrm{mod}_{fg} & \xrightarrow{\text{restr'n}} & Z^{\Xi_2}\text{-}\mathrm{mod}_{fg} \end{array}$$

2)
$$\begin{array}{ccc} \mathcal{O}_A^{\Xi_1} & \xleftarrow{T_{\lambda_2\to\lambda_1}} & \mathcal{O}_A^{\Xi_2} \\ \downarrow \hat{\mathbb{V}}_{\lambda_1} & & \downarrow \hat{\mathbb{V}}_{\lambda_2} \\ Z^{\Xi_1}\text{-}\mathrm{mod}_{fg} & \xleftarrow{A^{W_1}\otimes_A A^{W_2}\,?} & Z^{\Xi_2}\text{-}\mathrm{mod}_{fg} \end{array}$$

1.2) Proof of 1)

Recall that $\underline{\hat{\mathbb{V}}}_{\lambda_i}$ denotes the composition of $\hat{\mathbb{V}}_{\lambda_i}$ with the pullback $Z^{\Xi_i}\text{-}\mathrm{mod}_{fg} \to A\text{-}\mathrm{mod}_{fg}$ under $A \hookrightarrow Z^{\Xi_i}$

We first establish iso $\underline{\hat{\mathbb{V}}}_{\lambda_2} \circ T_{\lambda_1\to\lambda_2} \simeq \underline{\hat{\mathbb{V}}}_{\lambda_1}$

We have $\underline{\hat{\mathbb{V}}}_{\lambda_2} \circ T_{\lambda_1\to\lambda_2} = \mathrm{Hom}_{\mathcal{O}_A}(P_A(\lambda_2^-), T_{\lambda_1\to\lambda_2}(?)) \xrightarrow{\sim} [\text{adj'n}]$
$\mathrm{Hom}_{\mathcal{O}_A}(T_{\lambda_2\to\lambda_1} P_A(\lambda_2^-), ?)$. So we need an isomorphism

$$T_{\lambda_2\to\lambda_1} P_A(\lambda_2^-) \xrightarrow{\sim} P_A(\lambda_1^-) \qquad (1)$$

After specializing to $\mathbb{C}$ this is Proposition in Sec 1.1 of

Lec 10. Then (1) follows as in the proof of Lemma in Sec 1.4 of Lec 11 using Sec 2.3 of Lec 8, details are an exercise.

As any homomorphism in $\mathcal{O}_A^{\Xi}$, (1) is Z^{Ξ_1}-equivariant. On the other hand Z^{Ξ_2} acts on $P_A(\lambda_2^-)$ and hence, by functoriality on the l.h.s. of (1). We get a homomorphism

$$Z^{\Xi_2} = End_{\mathcal{O}_A}(P_A(\lambda_2^-)) \to End_{\mathcal{O}_A}(P_A(\lambda_1^-)) = Z^{\Xi_1}.$$

Lemma: This homomorphism coincides with the embedding $Z^{\Xi_2} \hookrightarrow Z^{\Xi_1}$ from Sec 1.1.

Proof: We have the inclusion $Z^{\Xi_i} \hookrightarrow A^{\oplus \Xi_i}$ that can be described as follows (Sec 1.4 of Lec 11): we have a filtration $\{0\} = P_0^i \subset P_1^i \subset \dots \subset P_{r_i}^i = P_A(\lambda_i^-)$ $(r_i = |\Xi_i|)$, it's preserved by $\forall\, z \in Z^{\Xi_i} \left(\xrightarrow{\sim} End_{\mathcal{O}_A}(P_A(\lambda_i^-))\right)$ & the embedding above is given by $z \mapsto (z_{M_j^i})_{j=1,\dots r_i}$, $M_j^i = P_j^i / P_{j-1}^i$ are all distinct Verma's in $\mathcal{O}_A^{\Xi_i}$.

Using the isomorphism $P_A(\lambda_1^-) \simeq T_{\lambda_2 \to \lambda_1} P_A(\lambda_2^-)$ we can induce the filtration on $P_A(\lambda_1^-)$ w. successive quotients $T_{\lambda_2 \to \lambda_1} M_j^2$; Z^{Ξ_2} acts on the latter by $z \mapsto z_{M_j^2}$. Each

$T_{\lambda_2\to\lambda_1} M_j^2$ is filtered by Vermas: $\Delta(w\cdot\lambda_1^+)$ s.t. $\Delta(w\cdot\lambda_2^+) \simeq M_j^2$. So we get the induced filtration on $P_A(\lambda_1^-)$, & use that filtration to define the embedding $Z^{\Xi_1}\hookrightarrow A^{\oplus \Xi_1}$.

We arrive at the following commutative diagram.

$$\begin{array}{ccccc} Z^{\Xi_2} & \hookrightarrow & A^{\oplus W/W_2} & \ni & (a_{wW_2})_{wW_2\in W/W_2} \\ \downarrow & & \downarrow & & \downarrow \leftarrow \text{denote this by } \gamma \\ Z^{\Xi_1} & \hookrightarrow & A^{\oplus W/W_1} & \ni & (a_{wW_2})_{wW_1\in W/W_1} \end{array}$$

Since the horizontal arrows are embeddings, the commutativity of the diagram recovers $Z^{\Xi_2}\hookrightarrow Z^{\Xi_1}$.

Recall, Step 2 in Sec 1.5 of Lec 12, that we have a commutative diagram:

$$\begin{array}{ccc} Z^{\Xi_i} = A^{W_i}\otimes_{A^W} A & \hookrightarrow & A^{\oplus W/W_i} \\ \downarrow & & \downarrow \leftarrow \text{spec. case of } \gamma \\ A\otimes_{A^W} A & \hookrightarrow & A^{\oplus W} \end{array}$$

This yields the following commutative diagram that finishes the proof:

$$\begin{array}{ccc} A^{W_2}\otimes_{A^W} A & \hookrightarrow & A^{\oplus W/W_2} \\ \downarrow & & \downarrow \gamma \\ A^{W_1}\otimes_{A^W} A & \hookrightarrow & A^{\oplus W/W_1} \end{array}$$

□

1.3) Proof of 2)

Set $\tilde{E}_i := \mathrm{End}_{\mathcal{O}_A}(\tilde{P}_A^i)^{opp}$, where $\tilde{P}_A^i$ is our usual projective generator of $\mathcal{O}_A^{\Xi_i}$ so that $\mathcal{O}_A^{\Xi_i} \xrightarrow{\sim} \tilde{E}_i\text{-mod}_{fg}$. Let $\varepsilon_i \in \tilde{E}_i$ be an idempotent with $P_A(\lambda_i^-) \longleftrightarrow \tilde{E}_i\tilde{\varepsilon}_i$ so that $\varepsilon_i\tilde{E}_i\varepsilon_i =: E_i = \mathrm{End}_{\mathcal{O}_A}(P_A(\lambda_i^-))^{opp}$. The functor $\hat{\mathbb{V}}_{\lambda_i}$ becomes

$$\pi_i : M_i \mapsto \varepsilon_i M_i = \mathrm{Hom}_{\tilde{E}_i}(\tilde{E}_i\varepsilon_i, M_i) : \tilde{E}_i\text{-mod}_{fg} \longrightarrow E_i\text{-mod}_{fg}$$

So it admits a left adjoint $\pi_i^* : N_i \mapsto \tilde{E}_i\varepsilon_i \otimes_{E_i} N_i$

Exercise: $\pi_i\pi_i^* \simeq id$.

Proof of 2): Set $\varphi = T_{\lambda_1 \to \lambda_2} : \tilde{E}_1\text{-mod}_{fg} \longrightarrow \tilde{E}_2\text{-mod}_{fg}$ & let $\underline{\varphi}$ denote the pullback functor $E_1\text{-mod}_{fg} \to E_2\text{-mod}_{fg}$

Then the left adjoints are $\varphi^* = T_{\lambda_2 \to \lambda_1}$ & $\underline{\varphi}^* = A^{W_1} \otimes_{A^{W_2}}$?

So 1) becomes a commutative diagram

$$\begin{array}{ccc} \tilde{E}_1\text{-mod}_{fg} & \xrightarrow{\varphi} & \tilde{E}_2\text{-mod}_{fg} \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ E_1\text{-mod}_{fg} & \xrightarrow{\underline{\varphi}} & E_2\text{-mod}_{fg} \end{array}$$

Pass to left adjoints: $\varphi^*\pi_2^* \simeq \pi_1^*\underline{\varphi}^* \Rightarrow$

$\pi_1\varphi^*\pi_2^*\pi_2 \simeq \pi_1\pi_1^*\underline{\varphi}^*\pi_2 \simeq$ [exercise] $\underline{\varphi}^*\pi_2$. 2) claims

that $\mathfrak{T}_1\varphi^* \simeq \varphi^*\mathfrak{T}_2$. It remains to show that the adjunction counit $\mathfrak{T}_2^*\mathfrak{T}_2 \xrightarrow{R} id$ induces $\mathfrak{T}_1\varphi^*\mathfrak{T}_2^*\mathfrak{T}_2 \xrightarrow{\sim} \mathfrak{T}_1\varphi^*$. Recall that $\mathfrak{T}_2(R) \simeq id$ so ker & coker of R are killed by $\mathfrak{T}_2$. So we reduce to showing:

$$\mathfrak{T}_2 M = 0 \Rightarrow \mathfrak{T}_1\varphi^* M = 0 \qquad (2)$$

$$\mathfrak{T}_1\varphi^* M = \mathrm{Hom}_{\mathcal{O}_A}(P_A(\lambda_1^-), \mathcal{T}_{\lambda_2\to\lambda_1} M) \simeq \mathrm{Hom}_{\mathcal{O}_A}(\mathcal{T}_{\lambda_1\to\lambda_2} P_A(\lambda_1^-), M)$$

In Sec 1.1 of Lec 10 we've seen $\mathcal{T}_{\lambda_1\to\lambda_2} P_{\mathbb{C}}(\lambda_1^-) \simeq P_{\mathbb{C}}(\lambda_2^-)^{\oplus ?}$

$\Rightarrow$ [Sec 2.3 of Lec 8; exercise] $\mathcal{T}_{\lambda_1\to\lambda_2} P_A(\lambda_1^-) \simeq P_A(\lambda_2^-)^{\oplus ?}$.

This proves (2) and finishes the proof $\square$

2) Applications to Soergel bimodules

We now revisit results in Lec 2. Consider the principal block $\mathcal{O}^{pr}$ & its deformed version $\mathcal{O}_A^{pr}$. For any word $\underline{w} \in S^k$ we have projectives $\textcircled{H}_{\underline{w}} \Delta(0) \in \mathcal{O}^{pr}$ & $\textcircled{H}_{\underline{w}} \Delta_A(0) \in \mathcal{O}_A^{pr}$, where for $\underline{w} = (s_1, \dots s_k)$, $\textcircled{H}_{\underline{w}} = \textcircled{H}_{s_k} \dots \textcircled{H}_{s_1}$ (cf. Sec 2 of Lec 7)

Lemma: Let $\underline{w}$ be a reduced expression of $w \in W$. Then

$$\textcircled{H}_{\underline{w}} \Delta_A(0) = P_A(w\cdot 0) \oplus \bigoplus_{l(u)<l(w)} P_A(u\cdot 0)^{\oplus ?}$$

Proof: By Thm in Sec 2.2 of Lec 7, $\bigoplus_{\underline{w}} \Delta_A(0) \twoheadrightarrow \Delta_A(w\cdot\lambda)$ w. ker filtered by $\Delta_A(u\cdot 0)$ w. $\ell(u)<\ell(w)$. Using $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}_A}(\Delta_A(\mu), L_{\mathbb{C}}(\lambda)) = \delta_{\lambda\mu}$ & Sec 2.3 of Lec 8 we see that $P_A(\lambda)$ occurs in $\bigoplus_{\underline{w}} \Delta_A(0)$ $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}_A}(\bigoplus_{\underline{w}} \Delta_A(0), L_{\mathbb{C}}(\lambda))$ times finishing the proof. □

Of course, the direct analog of Lemma holds for $\bigoplus_{\underline{w}} \Delta_{\mathbb{C}}(0)$

Consider the completion $\hat{BS}_{\underline{w}} = A\otimes_{A^{s_1}} A\otimes \dots \otimes_{A^{s_k}} A$ of the Bott-Samelson bimodule BS_w from Sec 1.1 of Lec 2.

By Thm in Sec 1.1,

$$\hat{\mathbb{V}}_0\left(\bigoplus_{\underline{w}} \Delta_A(0)\right) \simeq \hat{BS}_{\underline{w}} \quad \forall\ \underline{w}\in S^k$$

The functors $\mathbb{V}_0$ are fully faithful on projectives (Thm in Sec 1 of Lec 9). Same holds for $\hat{\mathbb{V}}_0$ thx to Proposition in Sec 1.1 of Lec 12. We deduce the following (exercise):

Corollary: 1) $\forall\ w\in W$, $\exists!$ indecomposable $A\otimes_{A^w} A$-module summand, $\hat{B}_w$, of $\hat{BS}_{\underline{w}}$ that doesn't appear in $\hat{BS}_{\underline{u}}$ w. $\ell(\underline{u})<\ell(w)$.

2) Any $\hat{BS}_{\underline{w}}$ is the direct sum of $\hat{B}_u$'s for $u \in W$.

3) $\hat{B}_u \otimes_A \mathbb{C}$ is an indecomposable A-module.

Using this we can deduce the following property of Soergel (6i) modules, Proposition in Sec 1.1 of Lec 2.

Prop 1) $\forall\ B_1, B_2 \in SBim(W) \Rightarrow H := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{SBim}(B_1, B_2\langle i \rangle)$ is free as right R-module &

$$H \otimes_R \mathbb{C}_0 \simeq \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{SMod}(B_1 \otimes_R \mathbb{C}, B_2\langle i \rangle \otimes_R \mathbb{C}) \qquad (3)$$

2) $\forall\ B \in SBim(W)$, B is indecomposable $\Leftrightarrow$ $B \otimes_R A$ is indecomposable in A-bimod (where $R = \mathbb{C}[\mathfrak{h}^*]$)

3) $M \in SMod(W)$ is indecomposable in $SMod(W) \Leftrightarrow$ it's indecomposable as an R-module (equiv. A-module).

Proof:

1) Every Soergel bimodule is a direct summand in a $\oplus$ of $BS_{\underline{w}}$ ($\underline{w} \in S^k$) so we can assume $B_i := BS_{\underline{w}_i}$.

Since B_1 is finitely generated as a bimodule, $H = \mathrm{Hom}_{R \otimes R}(B_1, B_2)$. And since B_1 is finitely generated as a

right R-module, $H \otimes_R A \xrightarrow{\sim} \mathrm{Hom}_{A\otimes A}(\hat{BS}_{\underline{w}_1}, \hat{BS}_{\underline{w}_2})$. The r.h.s. is $\mathrm{Hom}_{\mathcal{O}_A}(\textcircled{H}_{\underline{w}_1}\Delta_A(0), \textcircled{H}_{\underline{w}_2}\Delta_A(0))$. It's free as a right A-module $\Rightarrow$ [exercise] H is free/R. Changing the base to $\mathbb{C}$ we get $\mathrm{Hom}_{\mathcal{O}_0}(\textcircled{H}_{\underline{w}_1}\Delta_0(0), \textcircled{H}_{\underline{w}_2}\Delta_0(0))$, which is identified w. $\mathrm{Hom}_R(BS_{\underline{w}_1}\otimes_R \mathbb{C}, BS_{\underline{w}_2}\otimes_R \mathbb{C})$, which is the r.h.s of (3).

A key observation to prove (2) & (3) is as follows.

Claim: Let E be a finite dimensional associative $\mathbb{C}$-algebra. Suppose that E is $\mathbb{Z}$-graded. If E has a nontrivial idempotent, then it has one in deg 0.

Proof:

Thx to lifting of idempotents (Sec 2.3 of Lec 8), it's enough to replace E w. $E/\mathrm{rad}(E) \simeq \bigoplus_{i=1}^{k} \mathrm{Mat}_{n_i}(\mathbb{C})$, and the condition of Claim means $k>1$ or $n_1>1$. Note that $\mathrm{Aut}(E)^\circ \xrightarrow{\sim} \prod_{i=1}^{k} PGL_{n_i}$ & a grading on E corresponds to an algebraic group homomorphism $\mathbb{C}^\times \to \mathrm{Aut}(E)^\circ$. Any such lifts to $\mathbb{C}^\times \to \prod_{i=1}^{k} GL_{n_i}$ (exercise) & if $k>1$ or $n_1>1$, E contains a deg 0 nontrivial idempotent. □

3) immediately follows from Claim. To prove 2) note that if $B \otimes_R \mathbb{C}$ is indecomposable, then so is $B \otimes_R A$, then so is B (exercise). Conversely, suppose $B \otimes_R \mathbb{C}$ is indecomposable, pick a deg 0 idempotent in $\mathrm{End}_R(B \otimes_R \mathbb{C})$. We can lift it to a family of compatible deg 0 idempotents in the graded algebra $\mathrm{End}_{R \otimes R}(B) \otimes_R A/\mathfrak{m}^k$, where $\mathfrak{m} \subset A$ is the maximal ideal, thx to 1). Now note that the grading on $\mathrm{End}_{R \otimes R}(B)$ is bounded from below & $\mathrm{End}_{R \otimes R}(B)$ coincides w.

$$\varprojlim_k \mathrm{End}_{R \otimes R}(B) \otimes_R A/\mathfrak{m}^k$$

in the category of <u>graded</u> algebras (the limit is taken component-wise). This yields a nontrivial deg 0 idempotent in $\mathrm{End}_{R \otimes R}(B)$ & shows that B is decomposable as a graded bimodule $\square$