O & Soergel, lecture 14: What's next?

1) Soergel theory afterparty

2) Generalizations of O

1) Soergel theory afterparty

1.1) Where else?

Soergel's approach (which, in essence, starts w. proving an equivalence of additive categories) has been successfully used in many different contexts.

- The work of Beilinson-Ginzburg-Soergel on the parabolic-singular Koszul duality. This has been generalized by Bezrukavnikov-Yun to Kac-Moody groups -which has been then carried over to the modular setting by Achar-Makisumi-Riche-Williamson. It also has been generalized by Bezrukavnikov-Vilonen to the case of quasi-split real groups.

- The work of Bezrukavnikov-Riche on the modular

version of Bezrukavnikov's equivalence of two realizations of the affine Hecke category. We'll discuss the complex version of this equivalence later.

- The work of I.L. & Webster on the uniqueness of tensor products of minimal categorical representations of Kac-Moody algebras.
- The work of Campbell-Dhillon-Raskin on a version of the Fundamental Local Equivalence in local quantum Geometric Langlands
- The work of Lusztig-Yun (finite type), Li & Dhillon-Li-Yun-Zhu (affine type) on endoscopy for affine Hecke categories.

1.2) Essential features - to be explored later.

1) In our approach we paid special attention to objects filtered by Verma's. These will continue to be very important.

2) It was very important to work with deforma-

tions over A - all versions of Soergel's approach to category $\mathcal{O}$ do this - but we placed a special emphasize on this. One reason why this is useful that we can base change, say to localizations - and get categories that are easier to understand. For example, we've basically proved that the base change to $\mathrm{Frac}(A)$ is semisimple (exercise: deduce this from Proposition 1 in Sec 1.1 of Lec 11). For height 1 prime ideals $\mathfrak{p} \subset A$, the base change $\mathcal{O}_{A_\mathfrak{p}}$ may become somewhat more interesting: if $\mathfrak{p} = (\alpha^\vee)$, $\mathcal{O}_{A_\mathfrak{p}}$ splits into the sum of blocks w. one or two Vermas (corresponding to $(S_\alpha, \cdot)$-orbits in Λ), see Remark in Sec 1.2 of Lec 11). We'll later see that these blocks have a fairly easy structure.

3) We have a functor $\underline{\mathbb{V}}_\lambda : \mathcal{O}_A^\Xi\text{-mod} \to A\text{-mod}_{fg}$ that automatically upgrades to $\mathbb{V}_\lambda : \mathcal{O}_A^\Xi \to Z(\mathcal{O}_A^\Xi)\text{-mod}_{fg}$. One can:

(i) Compute $Z(\mathcal{O}_A^\Xi)$ as a subalgebra in $A^{\oplus \Xi}$ (Sec 1.3 of Lec 12).

(ii) Compute $\mathbb{V}_\lambda(\Delta_\Lambda(w\cdot\lambda^+))$ (essentially, Sec 2.3 of Lec 10) as a module over this algebra.

(iii) Establish certain faithfulness properties of $\hat{\mathbb{V}}_\lambda$ on Verma filtered objects and their specializations (hasn't been addressed explicitly yet).

1.3) Essential features that are too restrictive for us.

We started w. a basic object in the additive category of projectives, $\Delta_\Lambda(\lambda^+)$, and then produced more objects using easy to understand functors: $\textcircled{H}_i$'s (for dominant λ^+) & more general translation functors. Both the image of the initial object under $\hat{\mathbb{V}}_\lambda$ & the interplay of these functors are easy to understand (cf. Lec 13). These properties are not present in several important applications (say to quantum categories $\mathcal{O}$).

1.4) What we are going to study?

We've described $\mathcal{O}_\Lambda^{pr}$-proj as an additive category in

commutative algebra/combinatorics terms (one can similarly describe $\mathcal{O}_A^{\Xi}$-proj for general Ξ). What we want is a similar in spirit description of the "exact category" $\mathcal{O}_A^{\Xi,\Delta}$ of all objects w. Verma filtrations. The description – that applies in other contexts as well – is in terms of "sheaves on moment graphs" that also appear in a geometric context (and were introduced in this context by Braden and Macpherson).

2) Generalizations of $\mathcal{O}$.

2.1) Quantum category $\mathcal{O}$.

For simplicity (of dealing with the quantum group) we restrict to the case of $\mathfrak{g}=\mathfrak{sl}_2$.

First consider the Drinfeld-Jimbo quantum group $U_v(\mathfrak{g})$ over $\mathbb{C}(v)$ – it's generated by $E, F, K^{\pm 1}$ w. relations:

$$KEK^{-1} = v^2 E$$

$$KFK^{-1} = v^{-2} F$$

$$EF - FE = \frac{K-K^{-1}}{v-v^{-1}}.$$

There are several commonly used $\mathbb{C}[v^{\pm 1}]$-lattices of $U_v(\mathfrak{g})$ (i.e. subalgebras A w. $\mathbb{C}(v)\otimes_{\mathbb{C}[v^{\pm 1}]} A = U_v(\mathfrak{g})$). We will need the so called "mixed" form: it's generated by $F, K^{\pm 1}$ & $E^{(n)}$ $n \geq 1$. Here $E^{(n)}$ is the quantum divided power: set $[n] = \frac{v^{n+1} - v^{-(n+1)}}{v - v^{-1}} = v^n + v^{n-2} + \ldots + v^{-n}$, $[n]! = \prod_{i=1}^{n} [i]$ & $E^{(n)} = E^n/[n]!$ Denote the mixed form by $U_v^{mix}(\mathfrak{g})$.

We note that $U_v(\mathfrak{g})$ is graded w. $\deg K^{\pm 1} = 0$, $\deg E = 1$, $\deg F = -1$; $U_v^{mix}(\mathfrak{g})$ is a graded subalgebra ($\deg v = 0$)

For $q \in \mathbb{C}^\times$, set $U_q^{mix}(\mathfrak{g}) := \mathbb{C}_q \otimes_{\mathbb{C}[v^{\pm 1}]} U_v^{mix}(\mathfrak{g})$.

Definition: The quantum category $\mathcal{O}$, to be denoted by $\mathcal{O}_0^q$ is the category of $\mathbb{Z}$-graded $U_q^{mix}(\mathfrak{g})$-modules, $M = \bigoplus_{i \in \mathbb{Z}} M_i$ s.t.

1) M is finitely generated over $U_q^{mix}(\mathfrak{g})$
2) K acts on M_i by q^i:
3) $\{i \mid M_i \neq 0\}$ is bounded from above.

Exercise 1: $\forall M \in \mathcal{O}_0^q$ is finitely generated over $\mathbb{C}[F]$ so

O_0^q is abelian.

Example (Verma module): To $n \in \mathbb{Z}$ we assign the Verma module $\Delta_0^q(n) = U_q^{mix}(\mathfrak{g})/U_q^{mix}(\mathfrak{g})(K - q^n, E^{(k)}(k>0))$ with coset of 1 in deg n.

Exercise 2: $\mathrm{Hom}_{O_0^q}(\Delta_0^q(n), M) \xrightarrow{\sim} \{m \in M_n \mid E^{(k)} m = 0 \ \forall k>0\}$

We will be interested in the case when q is a root of 1 of odd order ℓ (the even roots of 1 are structurally more complicated). In this case, O_0^q acquires a new feature: an action of $\mathbb{Z}$ by autoequivalences: $1 \in \mathbb{Z}$ shifts the grading by ℓ: $(M\langle 1\rangle)_k := M_{k-\ell}$. Note that $\Delta_0^q(n)\langle 1\rangle = \Delta_0^q(n+\ell)$.

Premium exercise:

1) The quantum Casimir $C = FE + \frac{qK - q^{-1}K^{-1}}{(q-q^{-1})^2}$ is central in $U_v(\mathfrak{g})$. Use this to produce an infinitesimal block decomposition of O_0^q so that $\Delta_0^q(n_1), \Delta_0^q(n_2)$ lie in the same block $\Leftrightarrow n_1 - n_2 \equiv 0 \bmod \ell$ or $n_1 + n_2 + 2 \equiv 0 \bmod \ell$.

2) Pick $n \in \mathbb{Z}$ & let $r \in \{1, \dots \ell\}$ be s.t. $n+1 \equiv r \bmod \ell$. Let $v_n \in \Delta_0^q(n)$ denote a highest weight vector. Then $E^{(k)} F^r v_n = 0$ yielding a homomorphism $\Delta_0^q(n-2r) \to \Delta_0^q(n)$. It's injective.

So each Verma in the block is included into next so there's no distinguished Verma (like $\Delta(\lambda^+)$ or $\Delta(\lambda^-)$ in $\mathcal{O}_0^{\Xi}$). Furthermore, $\mathcal{O}_0^q$ doesn't have projective objects (but contains fairly complicated projective proobjects). Hence the essential features of the Soergel theory explained in Sec 1.3 are not present. Still, an approach hinted in Sec 1.4 can be applied to describe the category of Verma filtered objects in a deformed version of $\mathcal{O}_0^q$ (two-parametric: we deform q & the action of K) & prove a version of Gaitsgory's conjecture, which is a derived equivalence of this & another category.

Rem: There is an analog of $\mathcal{O}_0^q$ (& its deformation) for

other simple Lie algebras w. similar structural features.

2.2) Affine category $\mathcal{O}$.

Here's another important version. Assume $\mathfrak{g}$ is simple. We normalize an invariant symmetric form $(\cdot,\cdot)$ on $\mathfrak{g}$ by $(\alpha^\vee,\alpha^\vee)=2$ if short coroot α.

We then consider the (polynomial) loop algebra $\mathfrak{g}[t^{\pm 1}]$ (w. bracket $[x_1f_1, x_2f_2]=[x_1,x_2]f_1f_2$, $x_i\in\mathfrak{g}$, $f_i\in\mathbb{C}[t^{\pm 1}]$) & its central extension: $\hat{\mathfrak{g}}=\mathfrak{g}[t^{\pm 1}]\oplus\mathbb{C}c$ with bracket $[x_1t^{n_1}, x_2t^{n_2}]=[x_1,x_2]t^{n_1+n_2}+n_1\delta_{n_1,-n_2}(x_1,x_2)c$.

The algebra $\hat{\mathfrak{g}}$ admits a triangular decomposition as follows: define the Cartan subalgebra $\mathfrak{h}^a=\mathfrak{h}\oplus\mathbb{C}c$ as well as analogs of $\mathfrak{h}^\pm$

$$\mathfrak{i}^0=\mathfrak{h}^+\oplus t\mathfrak{g}[t], \quad \mathfrak{i}^{0,-}=\mathfrak{h}^-\oplus t^{-1}\mathfrak{g}[t^{-1}].$$

Consider the affine root lattice Λ^a with basis $\alpha_1,\dots\alpha_n$ – the simple roots for $\mathfrak{g}$ – and an additional element δ (the indecomposable imaginary root). Note $\hat{\mathfrak{g}}$ is Λ^a-graded w. $\deg c=0$, $\deg \mathfrak{g}_\alpha t^m=\alpha+m\delta$, $\deg \mathfrak{h}t^m=m\delta$. The degrees

occuring in $\mathfrak{i}$ ($\alpha+m\delta$ for $m>0$, here $\alpha\in$ roots of $\mathfrak{g}$ or 0; or $\alpha\in$ positive roots of $\mathfrak{g}$) are called the positive affine roots. Note that there is a (degenerate!) pairing

$$\Lambda^a \times \mathfrak{h}^a \longrightarrow \mathbb{C} : \langle \alpha+m\delta, x+zc\rangle = \langle \alpha, x\rangle \quad (x\in\mathfrak{h}, z\in\mathbb{C}) \qquad (1)$$

Definition: Let $\kappa\in\mathbb{C}$ (the level)

By the affine category $\mathcal{O}_\kappa^a$ we mean the category of Λ^a-graded $\hat{\mathfrak{g}}$-modules $M=\bigoplus_{\lambda^a\in\mathfrak{h}^a} M_{\lambda^a}$ s.t.

- $\dim M_{\lambda^a}<\infty \ \forall\ \lambda^a\in\Lambda^a$
- c acts on M by κ, $x\in\mathfrak{h}$ acts on M_{λ^a} by $\langle\lambda^a, x\rangle$
- $\{\lambda^a \mid M_{\lambda^a}\neq 0\}$ is bounded from above ($\exists\ \lambda_0^a$ s.t. $M_{\lambda^a}\neq 0 \Rightarrow \lambda_0^a-\lambda^a$ is a sum of positive affine roots).
- $\dim M_{\lambda^a}<\infty \ \forall\ \lambda^a$

This is an abelian category.

Example (Verma modules). To $\lambda^a\in\Lambda^a$ we assign the Verma module $\Delta_\kappa(\lambda^a)=U(\hat{\mathfrak{g}})/U(\hat{\mathfrak{g}})(x-\langle\lambda^a,x\rangle, c-\kappa, y \mid x\in\mathfrak{h}, y\in\mathfrak{i}^\circ)$

Remark: In fact, Verma modules may fail to be Noetherian so restricting to finitely generated modules does not give an abelian category.