O & Soergel, lecture 15: What's next? 2

1) Affine category $\mathcal{O}$

Recall: the pairing

$$\Lambda^a \times \mathfrak{h}^a \to \mathbb{C} : \langle \alpha + m\delta, x + zc \rangle = \langle \alpha, x \rangle \quad (x \in \mathfrak{h}, z \in \mathbb{C}) \qquad (1)$$

Definition: Let $\kappa \in \mathbb{C}$ (the level)

By the affine category $\mathcal{O}^a_\kappa$ we mean the category of Λ^a-graded $\hat{\mathfrak{g}}$-modules $M = \bigoplus_{\lambda^a \in \mathfrak{h}^a} M_{\lambda^a}$ s.t.

- $\dim M_{\lambda^a} < \infty \ \forall \ \lambda^a \in \Lambda^a$
- c acts on M by κ, $x \in \mathfrak{h}$ acts on M_{λ^a} by $\langle \lambda^a, x \rangle$
- $\{\lambda^a \mid M_{\lambda^a} \neq 0\}$ is bounded from above ($\exists\ \lambda^a_0$ s.t. $M_{\lambda^a} \neq 0 \Rightarrow \lambda^a_0 - \lambda^a$ is a sum of positive affine roots).
- $\dim M_{\lambda^a} < \infty \ \forall \ \lambda^a$

Today we'll study $\mathcal{O}^a_\kappa$ in more detail

1.1) Sugawara operators T_j

Since (1) is degenerate, the grading on M is not reco-

vered from the $\mathfrak{h}$-action. However, for all R but one, we can essentially recover the grading from the $\hat{\mathfrak{g}}$-action.

Namely choose two dual bases $x_1, \dots, x_d$ & $x^1, \dots, x^d$ in $\mathfrak{g}$.

For $i \in \mathbb{Z}$, set:

$$T_j = \begin{cases} \sum_{m \in \mathbb{Z}} \sum_{i=1}^{d} (x_i t^{-m})(x^i t^{m+j}), & j \neq 0 \\ \sum_i x_i x^i + 2 \sum_{m>0} (x_i t^{-m})(x^i t^m), & j = 0 \end{cases}$$

T_0 is a well-defined operator on any module from $\mathcal{O}_R^a$.

Note that $\sum_i [x_i t^{-m}, x^i t^{m+j}] = -m \delta_{j,0} (\dim \mathfrak{g}) c$ (exercise), so T_j, $j \neq 0$, is also a well-defined operator. The following fact is very important in the study of category $\mathcal{O}$

Fact: 1) $[T_j, xt^\ell] = -2(c + h^\vee) \ell (x t^{j+\ell})$. Here $h^\vee$ is the dual Coxeter number, i.e. $\frac{1}{2} \frac{(\cdot,\cdot)_{Kil}}{(\cdot,\cdot)}$, where $(\cdot,\cdot)_{Kil}$ is the Killing form.

2) For $\lambda^a = \lambda + m\delta$, the element T_0 acts on the highest vector in $\Delta_R^a(\lambda^a)$ by $((\lambda+\rho, \lambda+\rho) - (\rho,\rho))$, where we write $(\cdot,\cdot)$ for the form on $\mathfrak{h}^*$ inverse to $(\cdot,\cdot)$ on $\mathfrak{h}$, it's normalized so that $(\alpha,\alpha) = 2$ for long roots α.

For $M = \bigoplus_{\lambda^a} M_{\lambda^a}$ define the operator ν_M acting on M_{λ^a} for $\lambda^a = \lambda + m\delta$ by $T_0 + 2(\kappa + h^\vee)m$. By 1) of Fact, ν_M is an endomorphism of M; by construction it's functorial (exercise). So $\mathcal{O}_\kappa^a = \bigoplus_{z \in \mathbb{C}} \mathcal{O}_\kappa^{a,z}$, where $\mathcal{O}_\kappa^{a,z}$ is the full subcategory of all objects M where z is the only eigenvalue of ν_M. By 2) of Fact, $\Delta_\kappa^a(\lambda + m\delta) \in \mathcal{O}_\kappa^{a,z}$ where $z = (\lambda+\rho, \lambda+\rho) - (\rho,\rho) + 2(\kappa + h^\vee)m$

1.2) Behavior depending on the level

There are 4 cases.

1) $\kappa \notin \mathbb{Q}$ is the easiest. The Vermas in $\mathcal{O}_\kappa^{a,z}$ are $\Delta(\lambda + m\delta)$ w. fixed m & $(\lambda+\rho, \lambda+\rho)$ (depending on z)

Let $z_0 = 2(\kappa + h^\vee)m$ for some m. Then $\bigoplus_{z \in z_0 + \mathbb{Z}} \mathcal{O}_\kappa^{a,z}$ is equivalent to $\mathcal{O}_0$ via $N \in \mathcal{O}_0 \mapsto U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t] \oplus \mathbb{C}c)} N$, where $t\mathfrak{g}[t]$ acts on N by 0 & c by κ.

For $\kappa \in \mathbb{Q}$, there are 3 different behaviors depending on the sign of $\kappa + h^\vee$. Essentially, they inherit dis-

joint features of the BGG category $\mathcal{O}$.

2) $\kappa+h^\vee \in \mathbb{Q}_{<0}$ (a.k.a. the negative level). For any $z\in\mathbb{Q}$ w. nonzero $\mathcal{O}_\kappa^{a,z}$, the poset of highest weights

$$\Lambda_z^a := \{\lambda+m\delta \mid z = (\lambda+\rho,\lambda+\rho)-(\rho,\rho)+2(\kappa+h^\vee)m\}$$

has the following property: $\forall\ \lambda^a\in\Lambda_z^a$ the set $\{\mu^a\in\Lambda_z^a \mid \mu^a\leq\lambda^a$ is finite (exercise). An argument similar to Sec 1 of Lec 6 shows that every object in $\mathcal{O}_\kappa^{a,z}$ has finite length (exercise). But $\mathcal{O}_\kappa^{a,z}$ has no projectives.

3) $\kappa+h^\vee\in\mathbb{Q}_{>0}$ (a.k.a. the positive level). Here the set $\{\mu^a\in\Lambda_z^a \mid \mu^a\geqslant\lambda^a\}$ is finite $\forall\ \lambda^a\in\Lambda_z^a$. Vermas have infinite length but one can show that $\mathcal{O}_\kappa^{a,z}$ has enough projectives that are filtered by Vermas.

4) Finally, $\kappa=-h^\vee$ (the critical level). Note that T_j's commute w. any element of $\hat{\mathfrak{g}}$ in this case. The sets Λ_z^a are not bounded in either direction.

1.3) Finer structure: non-critical level

It turns out that the categories $\mathcal{O}_\kappa^a$ admit analogs of the infinitesimal block decomposition refining the decomposition $\mathcal{O}_\kappa^a = \bigoplus_{z\in\mathbb{C}} \mathcal{O}_\kappa^{a,z}$.

For a root $\beta = \alpha + n\delta$ define the coroot $\beta^\vee = \alpha^\vee + nc$ & a reflection s_β acting on $\Lambda^a \otimes_{\mathbb{Z}} \mathbb{C}\oplus\mathbb{C}$:

$$s_\beta \cdot (\lambda + x\delta, z) = (\lambda + x\delta - \langle \beta^\vee, (\lambda + x\delta, z)\rangle^\cdot \beta, z)$$

where $\langle \beta^\vee, (\lambda + x\delta, z)\rangle^\cdot = \langle \alpha^\vee, \lambda + \rho\rangle + n(z + h^\vee)x$

In particular, $s_\beta \cdot$ preserves the quadratic form (exercise)

$$q: \Lambda^a \otimes_{\mathbb{Z}} \mathbb{C}\oplus\mathbb{C} \longrightarrow \mathbb{C},\ (\lambda + x\delta, z) \mapsto (\lambda+\rho, \lambda+\rho) + 2(z+h^\vee)x$$

The subgroup in the group of affine transformations on $\Lambda^a \otimes_{\mathbb{Z}} \mathbb{C}\oplus\mathbb{C}$ generated by s_β's is the affine Weyl group, to be denoted by W^a.

We introduce an equivalence relation $\sim_\kappa$ on Λ^a by $\lambda^a \sim_\kappa \mu^a$ if $(\lambda^a, \kappa), (\mu^a, \kappa)$ lie in the same W^a-orbit. The following result is due to Kac (1984) via a version of the HC 150.

Fact: Let $\kappa \neq -h^\vee$. Then $\mathcal{O}_\kappa^a$ decomposes as $\bigoplus_{\Xi} \mathcal{O}_\kappa^{a,\Xi}$, where

the summation is over the equivalence classes for $\sim_\kappa$ & the Vermas in $O_\kappa^{a,\Xi}$ are exactly $\Delta_\kappa^a(\lambda^a)$, $\lambda^a \in \Xi$

It turns out that (for $\kappa \neq -h^\vee$) the "infinitesimal blocks" $O_\kappa^{a,\Xi}$ admit Soergel theoretic descriptions. To state it, for simplicity assume that $\kappa \in \mathbb{Z}$ & Ξ is a free orbit.

We can form the category of Soergel modules for a natural action of W^a on $\mathfrak{h}^{a,*}$, denote it by $SMod(W^a)$

Let's start w. $\kappa > -h^\vee$. Then $O_\kappa^{a,\Xi}\text{-proj} \xrightarrow{\sim} SMod_{ugr}(W^a)$

If $\kappa < -h^\vee$, then instead of projectives on considers the so called "tilting modules" (to be defined later)

1.4) Finer structure: critical level.

Now let us explain what happens when $\kappa = -h^\vee$ (the critical level). Here a suitable version of

$$U_{crit} := U(\hat{\mathfrak{g}})/(c+h^\vee)$$

has a large center. To define this version consider the left ideals $I_n := U_{crit}(t^n\mathfrak{g}[t]) \subset U_{crit}$. The inverse limit

$$\tilde{U}_{crit} = \varprojlim_n U_{crit}/I_n$$

carries a natural structure of a topological algebra, it acts on all modules in $\mathcal{O}^a_{-h^\vee}$. The operators T_j ($j \in \mathbb{Z}$) can be viewed as elements of $\tilde{U}_{crit}$. They are central and, in fact, topologically generate $Z(\tilde{U}_{crit})$ if $\mathfrak{g} = \mathfrak{sl}_2$. In the general case, the center is (topologically) $\mathbb{N}$-graded by, where the grading is induced by deg in t (in particular, $\deg T_j = j$). We can consider the infinitesimal block decomposition of $\mathcal{O}^a_{-h^\vee}$ w.r.t. the subalgebra of deg 0 elements (in fact, only $\mathfrak{r}\mathfrak{k}\mathfrak{g}$ elements are going to matter, this includes T_0). The resulting decomposition is

$$\mathcal{O}^a_{-h^\vee} = \bigoplus_{\Xi \in \Lambda/(W,\cdot)} \mathcal{O}^{a,\Xi}_{-h^\vee},$$

where the Vermas in $\mathcal{O}^{a,\Xi}_{-h^\vee}$ are $\Delta^a_{-h^\vee}(\lambda + n\delta)$, $\lambda \in \Xi$, $n \in \mathbb{Z}$.

In particular, in the general case, the set of highest weights in $\mathcal{O}^{a,\Xi}_{-h^\vee}$ is "smaller" then the affine Weyl group. On the other hand, $\mathcal{O}^{a,\Xi}_{-h^\vee}$ is "too large" to be controlled by "Soergel theory", either more classical or the modification advertised in Sec 1.4. More precisely, for the

theory to work, Homs between Vermas should have dim ≤ 1. However, in the case of $\mathfrak{g}=\mathfrak{sl}_2$ one can show that

$$\mathrm{Hom}_{\mathcal{O}^a_{-h^\vee}}(\Delta^a_{-h^\vee}(\lambda^a), \Delta^a_{-h^\vee}(\lambda^a + m\delta))$$

has basis $T_{i_1}\dots T_{i_k}$ w. $0>i_1,\dots i_k$ & $\sum_{j=1}^{k} i_j = m$.

1.5) Blockwise Gaitsgory's conjecture

Let $\kappa+h^\vee \notin \mathbb{Q}_{\geq 0}$. Set $q=\exp(\pi\sqrt{-1}/(\kappa+h^\vee))$. Gaitsgory predicted an equivalence (in our notation) between:

- $D^b(\mathcal{O}_\kappa^{a,t})$ for suitable value of t
- & the full subcategory in $D^b(\mathcal{O}_0^a)$ of all objects w. weights in the root lattice. In the most interesting case $\kappa+h^\vee \in \mathbb{Q}_{<0}$ the equivalence is not t-exact. The Verma modules $\Delta_0^a(\lambda)$ correspond to so called **Waki-moto modules** $\mathrm{Wak}_\kappa(\lambda+\eta_\lambda\delta)$ (η_λ is recovered from λ & z). Gaitsgory's equivalence is supposed to satisfy some compatibilities. Various partial results on this were proved by L. Chen & Y. Fu, Q. Situ, and I.L.

An approach of I.L. is based on a version of what

was explained in Lec 14. It works block by block and places some restrictions on $\mathfrak{k}$ (q should be an odd root of 1, it should be possible to remove this). Key ingredients are deforming over $A := \mathbb{C}[[\mathfrak{h}^{a,*}]]$, constructing full embeddings of the categories of Verma & Wakimoto filtered objects into the same category ($A^{W_0} \otimes A\text{-mod}_{fg}$ for suitable reflection group W_0) via Soergel like functors & proving that the images coincide.

1.6) What's next?

While the equivalence in Sec 1.5 is the first serious application of the techniques advertised in Sec 1.4 of Lec 14. But this is not a motivation to develop them - which was a double affine version of Bezrukavnikov's equivalence. Our next task is to describe the usual version due to Bezrukavnikov. To bridge the categories $\mathcal{O}$ (quantum & affine) and this equivalence we will discuss Beilinson-Bernstein type localization theorems for

non-critical affine categories $\mathcal{O}$.