Q & Soergel, lec 16: What's next? 3

1) Bezrukavnikov's equivalence

This equivalence relates two categorical versions of the affine Hecke algebra: one via constructible sheaves/D-modules on an affine flag variety, the other via coherent sheaves on a Steinberg variety. This result is a very important special case of categorical local Langlands correspondence.

1.1) Overview of finite case.

For a connected reductive group G we write $\check{G}$ for its Langlands dual defined as follows. To an algebraic torus T we assign its character, $\mathfrak{X}^*(T)$, & co-character, $\mathfrak{X}_*(T)$, lattices: $\mathfrak{X}^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$, $\mathfrak{X}_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$.

For a maximal torus $H \subset G$ we have root & coroot systems: $\Delta_G \subset \mathfrak{X}^*(H)$ (=nonzero weights of $H \curvearrowright \mathfrak{g}$) & $\Delta_G^\vee \subset \mathfrak{X}_*(H)$. The group $\check{G}$ is characterized by the follo-

wing property, if $\check{H}$ denotes the maximal torus:

$$(\mathfrak{X}^*(H), \mathfrak{X}_*(H), \Delta_G, \Delta_G^\vee) = (\mathfrak{X}_*(\check{H}), \mathfrak{X}^*(\check{H}), \Delta_G^\vee, \Delta_G)$$

We'll need the following two facts:

1) Corollary in Sec 1.2 of Lec 2: If $\mathcal{O}^{pr}(\mathfrak{g})$ denotes the principal block of category $\mathcal{O}$ for $\mathfrak{g}$, then $\mathcal{O}^{pr}(\mathfrak{g}) \xrightarrow{\sim} \mathcal{O}^{pr}(\check{\mathfrak{g}})$ (an equivalence of abelian categories).

2) Let $\mathcal{D}(G)\text{-mod}_{fg}^{N\times B}$ denote the category of finitely generated (over $\mathcal{D}(G)$) $N\times B$-equivariant $\mathcal{D}(G)$-modules (where $N \subset B \subset G$ are Borel & maximal unipotent subgroups). Then we have an equivalence of abelian categories $\mathcal{O}^{pr}(\mathfrak{g}) \xrightarrow{\sim} \mathcal{D}(G)\text{-mod}_{fg}^{N\times B}$.

Combining 1) & 2) we get

Corollary: We have an equivalence of abelian categories $\mathcal{D}(G)\text{-mod}^{N\times B} \xrightarrow{\sim} \mathcal{D}(\check{G})\text{-mod}^{\check{N}\times\check{B}}$.

The categories in Bezrukavnikov's equivalence "affinize"

the two sides in two different ways.

1.2) Constructible side

Consider the loop group $G((t))$, an infinite dimensional algebraic group that for time being we view just as the $\mathbb{C}((t))$-points of G. Inside we have the subgroup $G[[t]]$ (of $\mathbb{C}[[t]]$-points of G) that comes w. a natural epimorphism $G[[t]] \xrightarrow[t=0]{} G$. Define the Iwahori subgroup $I \subset G[[t]]$ as the preimage of $B \subset G$. Let $I^0 \subset I$ denote the preimage of N in $G[[t]]$.

Here's an affine analog of the Bruhat decomposition $G = \coprod_{w \in W} NwB$. For $\nu \in \mathcal{X}_*(H)$ consider the element $t^\nu \in H((t))$ (e.g. if $G = GL_n$ then ν is encoded by $(\nu_1, \dots \nu_n) \in \mathbb{Z}^n$, then $t^\nu = \mathrm{diag}(t^{\nu_1}, \dots t^{\nu_n})$). Define the affine Weyl group W_G^a as $W \ltimes \mathcal{X}_*(T)$, and let τ_ν denote the image of ν in W_G^a. Then $\forall\ x = w\tau_\nu \in W_G^a$, the element $\dot{w} t^\nu$ (where $\dot{w} \in N_G(T)$ is a lift of w) normalizes $H \subset G((t))$

Fact (affine Bruhat decomposition)

1) $G((t)) = \bigsqcup_{x \in W_G^a} I^\circ \dot{x} I$.

2) $I^\circ \dot{x} I/I$ is an affine space of dimension equal to the length of x.

Premium exercise: Prove this for $G = GL_n$.

We have $\mathcal{D}(G)\text{-mod}^{N \times B} \xrightarrow{\sim} \mathcal{D}_{G/B}\text{-mod}^N$ (cf. Sec 2.1 of Lec 3).

Set $\mathcal{F}\ell = G((t))/I$, the affine flag variety. We can consider the derived category $D^b_{I^\circ}(\mathcal{F}\ell)$ of complexes of I°-equivariant $\mathcal{D}$-modules, where the total cohomology has finite length. It's generated by the following objects. Let $\imath_x : I^\circ \dot{x} I/I \hookrightarrow \mathcal{F}\ell$ be the locally closed inclusion, it's an affine morphism. We can consider the star, $\imath_{x,*} \mathcal{O}_{I^\circ \dot{x} I/I}$ (cf. Lec 4) & shriek $\imath_{x,!} \mathcal{O}_{I^\circ \dot{x} I/I}$ pushforwards.

Example: If $G = H$, then $\mathcal{X}_*(H) \xrightarrow{\sim} \mathcal{F}\ell$ via $\nu \mapsto t^\nu I$ so $\mathcal{F}\ell$ is discrete. I° is a prounipotent group, so $D^b_{I^\circ}(\mathcal{F}\ell) \simeq D^b(\mathcal{F}\ell)$ $\simeq$ bounded derived category of fin. dim $\mathcal{X}_*(H)$-graded vec-

tor spaces $\simeq [\mathcal{X}_*(H) \simeq \mathcal{X}^*(\check{H})]$ $\mathcal{D}^b \mathrm{Rep}(\check{H})$.

1.3) Coherent side

Now let us explain the "spectral side"

For a smooth affine variety, X, the algebra $\mathcal{D}(X)$ carries a filtration by the order of differential operator (w. $\deg \mathbb{C}[X]=0$, $\deg \mathrm{Vect}(X)=1$) & $\mathrm{gr}\,\mathcal{D}(X) \simeq \mathbb{C}[T^*X]$.

Now suppose an algebraic group F acts on X, and M be a strongly equivariant fin. generated $\mathcal{D}(X)$-module. We can find an F-stable good filtration $M = \bigcup_{j \geqslant 0} M_{\leqslant j}$ - compatible with the filtration on $\mathcal{D}(X)$ and s.t. $\mathrm{gr}\,M$ is a fin. generated $\mathrm{gr}\,\mathcal{D}(X)$-module. Since the filtration is F (& so $\mathfrak{f}$-) -stable & $\deg \xi_X = 1$ $\forall\, \xi \in \mathfrak{f}$, we see that $\xi_X \in \mathbb{C}[T^*X]$, $\xi \in \mathfrak{f}$, act by 0 on $\mathrm{gr}\,M$. So we see that $\mathrm{gr}\,M$ is an F-equivariant coherent sheaf on $\mu_F^{-1}(0)$, where $\mu_F : T^*X \longrightarrow \mathfrak{f}^*$ given by $\langle \mu_F(x,\alpha), \xi \rangle = \langle \alpha, \xi_{X,x} \rangle$ is the moment map. Our conclusion is that $\mathrm{gr}\,M$ is an

F-equivariant coherent sheaf on $\mu_F^{-1}(0)$.

Let's apply this to $X = \check{G}$, $F = \check{N} \times \check{B}$. We can identify $T^*\check{G}$ w. $\check{G} \times \check{\mathfrak{g}}^*$ by using right invariant vector fields. Then $\mu_F : (g, \alpha) \mapsto (\alpha|_{\check{\mathfrak{n}}}, -g.\alpha|_{\check{\mathfrak{b}}})$, so we are looking at $\check{N} \times \check{B}$-equivariant coherent sheaves on

$$\{(g, \alpha) \in \check{G} \times \check{\mathfrak{g}}^* \mid \alpha \in (\check{\mathfrak{g}}/\check{\mathfrak{n}})^* \cap g.(\check{\mathfrak{g}}/\check{\mathfrak{b}})^*\} \qquad (1)$$

Example: $\check{G} = \check{H} = \check{B}$, $\check{N} = pt$; (1) becomes $\check{H}$ & we are looking at $\check{H}$-equivariant coherent sheaves, so the resulting category is Vect.

This disagrees with Example in Sec 1.3. What we missed is that $\mathcal{D}(\check{G})\text{-mod}^{\check{N} \times \check{B}}$ has a natural <u>weakly</u> $\check{H}$-equivariant lift (w. $\check{H}$-acting from the left) so instead of $\check{N} \times \check{B}$-equivariant coherent sheaves on (1) we should have considered $\check{B} \times \check{B}$-equivariant ones.

Now note that that to give a point of (1) up to the $\check{B} \times \check{B}$-action is the same as to give a point of

$\check{St} := \{(g_1\check{B}, g_2\check{B}, \alpha) \in (\check{G}/\check{B})^2 \times \check{\mathfrak{g}}^* \mid \alpha \in g_1.(\check{\mathfrak{g}}/\check{\mathfrak{n}})^* \cap g_2(\check{\mathfrak{g}}/\check{\mathfrak{b}})^*\}$ up to the diagonal $\check{G}$-action; $\check{St}$ is known as the Steinberg scheme (it's not reduced) and the spectral side we need is $D^b Coh^{\check{G}}(\check{St})$.

1.4) The equivalence

Thm (Bezrukavnikov) We have an equivalence of triangulated categories $D^b_{I_0}(Fl) \xrightarrow{\sim} D^b Coh^{\check{G}}(St)$.

Remarks: 1) A starting point for this theorem is a classical result of Kazhdan-Lusztig & Ginzburg that the affine Hecke algebra for W_G^a is isomorphic to $K_0(Coh^{\check{G}\times \mathbb{G}_m}(\check{St}))$, where $\mathbb{G}_m$ acts on (1) by rescaling α. One should be able to upgrade Thm to recover the missing equivariance but this is not in the literature. Also note that the category in the equivalence are not monoidal (but they are "highest weight"). Bezrukavnikov also proves a monoidal version: one replaces $D^b_{I_0}(Fl)$ w.

$D^b_I(Fl)$ & $\check{St}$ w. the (derived) scheme

$\check{St}_0 = \{(g_1B, g_2B, \alpha) \in (\check{G}/\check{B})^2 \times \check{\mathfrak{g}}^* \mid \alpha \in g_1.(\check{\mathfrak{g}}/\check{\mathfrak{b}}_1)^* \cap g_2.(\check{\mathfrak{g}}/\check{\mathfrak{b}})^*\}$.

2) In this theorem we see two ways to "affinize":

a) Add a loop on the level of the space:

$\mathcal{D}_{G/B}\text{-mod}^N \rightsquigarrow \mathcal{D}_{G((t))/I}\text{-mod}^{I^\circ}$.

b) Degenerate: $\mathcal{D}_{\check{G}/\check{B}}\text{-mod}^{\check{N}} \rightsquigarrow Coh^{\check{G}}(\check{St})$ or on the level of stacks: $\mathcal{D}_{\check{N}\backslash\check{G}/\check{B}}\text{-mod} \rightsquigarrow Coh(T^*(\check{N}\backslash\check{G}/\check{B})/\check{H})$.

One should explain why this is "affinization". We have distinguished objects indexed by W in $\mathcal{D}_{\check{N}\backslash\check{G}/\check{B}}\text{-mod}$ (one can take $\imath_{w,!}\mathcal{O}, w \in W$, or $\imath_{w,*}\mathcal{O}, w \in W$). One can then equip them with suitable good filtrations and takes associated graded objects (technically, these objects admit distinguished mixed Hodge structures and one takes the Hodge filtrations). Then for $\lambda \in \mathfrak{X}^*(\check{H})$ we have the line bundles $\mathcal{O}(\lambda, 0)$ or $\mathcal{O}(0, \lambda)$ in $Coh^{\check{G}}(\check{St})$ (pulled back from G-equivariant line bundles on the 1st & 2nd copies of G/B) & we can consider objects like $(gr\,\imath_{w,*}\mathcal{O})(0, \lambda)$ or

$(\mathrm{gr}\, \iota_{w,!}\mathcal{O})(\lambda,0)$ – four options). So we again get families of objects labelled by W_G^a. Under the Bezrukavnikov equivalence they go to "Wakimoto sheaves" – objects intermediate between $\iota_{x,!}\mathcal{O}$ & $\iota_{x,*}\mathcal{O}$ (under suitable normalizations).

There are other instances of degeneration:

i) Passing from char 0 to char $p>>0$. For example, recall from Sec 2.1 that $\mathcal{O}_0$ is the category of strongly B-equivariant (finitely generated) $U(\mathfrak{g})$-modules. One can consider the similar category in characteristic p. For $p>h$ (the Coxeter number) its principal block is derived equivalent to $D^b\mathrm{Coh}^{G^{(1)}}(St^{(1)})$, where (1) is the Frobenius twist. This equivalence is useful for example in I.L.'s approach to block-wise Gaitsgory's conjecture.

ii) Passing from the quantum category $\mathcal{O}$ for generic q to q equal to a root of 1.