O & Soergel, lec 17: What's next? 4

1) Localization for affine category $\mathcal{O}$

2) Highest weight categories.

1) Localization for affine category $\mathcal{O}$

1.1) Negative level localization

Recall that $\mathcal{O}^{pr}(\mathfrak{g})$ (hence $\mathcal{O}^{\Xi}(\mathfrak{g}) \simeq \mathcal{O}^{pr}(\mathfrak{g})$ in the case when $\Xi \subset \Lambda/(W,\cdot)$ is a free orbit, cf. Sec 1 of Lec 8) is equivalent to $\mathcal{D}_{G/B}\text{-mod}^N$.

Consider the affine case. Suppose $\kappa \in \mathcal{K}_{<-h^\vee}$. Since $\kappa \in \mathcal{K}$, the $\sim_\kappa$-equivalence classes are W^a-orbits (see Sec 1.3 of Lec 15) and we pick a free orbit Ξ (which only exists if κ is sufficiently far from $-h^\vee$).

Next recall the category $D^b_{I^0}(\mathcal{FL})$, the affine analog of $D^b(\mathcal{D}_{G/B}\text{-mod}^N)$ from Sec 1.2 of Lec 16. The following is due to Kashiwara & Tanisaki.

Thm: There's a t-exact equivalence

$$D^b(\mathcal{O}_\kappa^{a,\Xi}) \xrightarrow{\sim} D^b_{I^0}(\mathcal{FL})$$

It sends Vermas to the objects $i_{x,!}\mathcal{O}_{I^0 x I/I}$.

1.2) Positive level localization

Now suppose $\kappa > -h^\vee$ & there's an equivalence class for $\sim_\kappa$ that is a free W^a-orbit. It turns out that there's a complete analog of Thm, also due to Kashiwara & Tanisaki, with one crucial difference: one replaces I (the preimage of $N \subset G$ in $G[[t]]$) with I^-, the preimage of the opposite Borel B^- in $G[t^{-1}] \subset G((t))$ under the projection $G[t^{-1}] \twoheadrightarrow G$.

It should be noted that the geometry of the action of $I^- \times I^0$ on $G((t))$ is very different from that for $I^- \times I^0$. Namely, $G((t))/I^-$ is an infinite type scheme, while $G((t))/I$ is an ind-projective ind-scheme (the union of projective schemes under closed embeddings).

The I^0-orbits on $G((t))/I$ are of finite dimension,

while the I^0-orbits on $G((t))/I^-$ are of finite codimension in a suitable sense. But in some ways the stack $I^- \backslash G((t))/I^0$ is nicer & easier than $I \backslash G((t))/I^0$:

I) It's covered by open substacks of the form X_i/F_i, where X_i is smooth variety & F_i is a finite dimensional algebraic group.

II) It has a nice moduli interpretation: the quotient $G[[t]] \backslash G((t))/G[t^{-1}]$ is the moduli stack of G-bundles on $\mathbb{P}^1$ so $I^- \backslash G((t))/I^0$ parameterizes G-bundles on $\mathbb{P}^1$ w. N-reduction at 0 & B^--reduction at ∞.

Example: Let $G = G_m$. Here we parameterize the line bundles on $\mathbb{P}^1$ trivialized at 0. We get a morphism $I^- \backslash G((t))/I^0 \to \mathbb{Z}$ sending a line bundle to its degree It's actually a stack isomorphism. This can be also seen group theoretically.

In contrast, $I^-\backslash G((t))/I^0$ is fairly complicated: $G((t))$ decomposes into the group product $I\times\mathbb{Z}\times I^{-,0}$ (e.g. by the previous paragraph) $I^-\backslash G((t))/I^0 \xrightarrow{\sim} \mathbb{Z}\times I^{-,0}\times pt/I^0$. $I^{-,0} = \operatorname{colim}_n$ (formal spectra of $\mathbb{C}[[x]]$).

The $\mathcal{D}$-modules don't care about this complications – they are of topological nature, cf. Example in Sec 1.2 of Lec 17.

Thanks to 1) one can define the category of $\mathcal{D}$-modules on $I^-\backslash G((t))/I^0$: on open pieces X_i/F_i we just get strongly equivariant $\mathcal{D}$-modules.

And we still have the affine Bruhat decomposition: $I^-\backslash G((t)) = \bigsqcup_{x\in W_G^a} I^-\backslash I^-xI^0$ (into locally closed subschemes)

The following is due to Kashiwara & Tanisaki

Approximate theorem: $\exists$ a t-exact equivalence

between (suitable completions of) derived categories of $\mathcal{O}_\kappa^{q,\Xi}$ & $\mathcal{D}\text{-mod}_{I^-\backslash G((t))/I^\circ}$ sending Vermas to the shriek extensions $i_{x,!}\mathcal{O}_{I^-\backslash I^-xI^\circ/I^\circ}$

Important remark*: While the (ind-)stacks $I\backslash G((t))/I^\circ$ & $I^-\backslash G((t))/I^\circ$ are quite different their categories of $\mathcal{D}$-modules are closely related: $\mathcal{D}_{I^\circ}(\mathcal{F}l)$ admits a full embedding into the derived category of $\mathcal{D}\text{-mod}_{I^-\backslash G((t))/I^\circ}$ sending $i_{x,*}\mathcal{O}_{I\backslash G((t))/I^\circ}$ to $i_{x,!}\mathcal{O}_{I^-\backslash G((t))/I^\circ}$. Namely we have the convolution functor:

$$\mathcal{D}(\mathcal{D}\text{-mod}_{I^-\backslash G((t))/I}) \boxtimes \mathcal{D}^b(\mathcal{D}\text{-mod}_{I\backslash G((t))/I^\circ}) \longrightarrow$$

$$\mathcal{D}(\mathcal{D}\text{-mod}_{I^-\backslash G((t))/I^\circ}) : \mathcal{F}\boxtimes\mathcal{G} \to \mathcal{F}*\mathcal{G}$$

and the full embedding is given by $\mathcal{G} \mapsto i_{1,!}\mathcal{O}*\mathcal{G}$.

This gives an example of "Ringel duality" - to be revisited later.

1.5) Toward double affine Bezrukavnikov equivalence

The Bezrukavnikov equivalence, $D_{I^0}(\mathcal{F}l) \to D^b Coh^{\check{G}}(\check{St})$ was obtained from $D(G)\text{-mod}^{N\times B} \xrightarrow{\sim} D(\check{G})\text{-mod}^{\check{N}\times\check{B}}$ by affinizing: we add a loop on the l.h.s. and degenerate on the r.h.s. It's not clear how to degenerate the 2nd time but one can, at least in principle, add loops several times, but already the 2nd time ("double affine") one gets a very complicated geometric object.

What is quite reasonable in this complicated world is the degeneration of $D\text{-mod}_{I^-\backslash G((t))/I^0}$ = +1 loop version of $D^b Coh^G(St)$ – a suitable version of the category of coherent sheaves on $T^*(I^-\backslash G((t))/I^0)$ (many modifications required). A deformation of this category is highest weight and is tractable to Soergel-theoretic techniques mentioned in Sec 1.4 of Lec 14.

Our results w. Davis suggest a strong connection between this category & the constructible category for the double affine flag variety. Interestingly,

for the usual affine flag variety what we get is not the Bezrukavnikov equivalence but its composition w. Koszul duality.

2) Highest weight categories.

2.1) Definition

Let A be a complete local Noetherian ring (e.g. $\mathbb{C}[[x_1,\dots x_n]]$) & let $\mathcal{C}_A$ be an A-linear abelian category w. Homs that are finitely generated over A.

Definition: A highest weight structure on $\mathcal{C}_A$ is:

- A poset $(\mathcal{T}, \leq)$
- & a collection of standard objects $\Delta_A(\tau) \in \mathcal{C}_A$ s.t.

(HW0) Each $\Delta_A(\tau)$ is flat over A.

(HW1) $\forall\, M \in \mathcal{C}_A \setminus \{0\}\ \exists\, \tau \in \mathcal{T}$ s.t. $\mathrm{Hom}_{\mathcal{C}_A}(\Delta_A(\tau), M) \neq 0$.

(HW2) $\mathrm{Hom}_{\mathcal{C}_A}(\Delta_A(\tau), \Delta_A(\tau')) \neq 0 \Rightarrow \tau \leq \tau'$

(HW3) $\mathrm{End}_{\mathcal{C}_A}(\Delta_A(\tau)) \xleftarrow{\sim} A \quad \forall\, \tau \in \mathcal{T}$

(HW4) $\forall\, \tau \in \mathcal{T}\ \exists\, P_\tau \in \mathcal{C}_A\text{-proj}$ w. $P_\tau \twoheadrightarrow \Delta_A(\tau)$ s.t. the kernel is filtered by $\Delta_A(\tau')$ w. $\tau' > \tau$.

Remark: Categories appearing in Lecs 14-17 (such as quantum & affine categories) are not highest weight

in the sense of this definition.

Exercise: For $i>0$, $\mathrm{Ext}^i_{\mathcal{C}_A}(\Delta_A(\tau), \Delta_A(\tau')) \neq 0 \Rightarrow \tau < \tau'$

2.2) Example

Let $A = \mathbb{C}[[\mathfrak{h}^*]]$ & $\mathcal{C}_A = \mathcal{O}_A^{\Xi}$. Equip $\mathcal{T} = \Xi$ w. the order restricted from Λ. For the standards we take the Verma modules $\Delta_A(\tau), \tau \in \mathcal{T}$. (HW0)-(HW3) are exercises.

To establish (HW4) we argue as follows. Set $\beta := \lambda^+ - \tau$. Note that $U(\mathfrak{k})$ is graded by the monoid $\mathrm{Span}_{\mathbb{Z}_{\geq 0}}(\alpha_1, \dots \alpha_r)$, where $\alpha_1, \dots \alpha_r$ are the simple roots. Consider the span $U(\mathfrak{k})_{\not\leq \beta}$ of all homogeneous elements whose weight is not $\leq \beta$. This a 2-sided ideal of finite codim.

Set $N_\tau := [U(\mathfrak{k})/U(\mathfrak{k})_{\not\leq \beta}] \otimes A$. This is naturally a $\mathfrak{b}$-module (w. $\mathfrak{h}$ acting according to the grading). Set

$$P_\tau := P_\Xi(U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} N_\tau) \in \mathcal{O}_A^{\Xi}$$

and let's check that (HW4) holds here:

- P_τ is projective in $\mathcal{O}_A^{\Xi}$, $M \in \mathcal{O}_A^{\Xi}$: $\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})\otimes A}(P_\tau, M) \xrightarrow{\sim}$ [$\otimes$ - Hom adj'n] $\mathrm{Hom}_{\mathcal{U}(\mathfrak{b})\otimes A}(N_\tau, M) \xrightarrow{\sim} \{m \in M_\tau \mid \mathcal{U}(\mathfrak{h})_{\not\leq \beta} m = 0\}$

But the weights in $\mathcal{U}(\mathfrak{h})_{\not\leq \beta} m$ are $\not\leq \lambda^+$, while all weights in modules in $\mathcal{O}_A^{\Xi}$ are $\leq \lambda^+$ (cf. Sec 2.1 in Lec 6). So $\mathcal{U}(\mathfrak{h})_{\leq \beta} m = 0$ & P_τ represents the functor $M \to M_\tau$, which is exact.

- One can find a complete flag of $\mathfrak{b}$-subreps in N_τ as in Sec 2.3 of Lec 6. This yields a filtration

$$\{0\} = P_0 \subset P_1 \subset P_2 \subset \ldots \subset P_r = P_\tau \text{ w. } P_i/P_{i-1} \simeq \Delta_A(\tau_i)$$

s.t. $\tau_r = \tau$ & $\tau_i > \tau$ if $i < r$. This establishes the required properties.