O&S Soergel, Lec 18: Highest weight categories.

1) Structure of highest weight categories.

1.0) Recap

Let A be a complete local Noetherian ring & let $\mathcal{C}_A$ be an A-linear abelian category w. finitely generated Homs. The following definition was given in Sec 2.1 of Lec 17.

Definition 1: A highest weight structure on $\mathcal{C}_A$ is:

- A poset $(\mathcal{T}, \leq)$
- & a collection of standard objects $\Delta_A(\tau) \in \mathcal{C}_A$ s.t.

(HW0) Each $\Delta_A(\tau)$ is flat over A.

(HW1) $\forall\, M \in \mathcal{C}_A \setminus \{0\}\ \exists\, \tau \in \mathcal{T}$ s.t. $\mathrm{Hom}_{\mathcal{C}_A}(\Delta_A(\tau), M) \neq 0$.

(HW2) $\mathrm{Hom}_{\mathcal{C}_A}(\Delta_A(\tau), \Delta_A(\tau')) \neq 0 \Rightarrow \tau \leq \tau'$

(HW3) $\mathrm{End}_{\mathcal{C}_A}(\Delta_A(\tau)) \xleftarrow{\sim} A \quad \forall\, \tau \in \mathcal{T}$

(HW4) $\forall\, \tau \in \mathcal{T}\ \exists\, P_\tau \in \mathcal{C}_A\text{-proj}$ w. $P_\tau \twoheadrightarrow \Delta_A(\tau)$ s.t. the kernel is filtered by $\Delta_A(\tau')$ w. $\tau' > \tau$.

In this lecture we will study some structural features of highest weight categories. Let's explain our main goal to be accomplished in later lectures.

Definition 2: An object $M \in \mathcal{C}_A$ is called standardly filtered if $\exists$ filtration: $\{0\} = M_0 \subset M_1 \subset \dots \subset M_r = M$ s.t. $M_i/M_{i-1} \simeq \Delta_A(\tau_i)$ $\forall$ $i = 1, \dots r$. The full subcategory of such objects is denoted by $\mathcal{C}_A^\Delta$

We'll see that $\mathcal{C}_A^\Delta$ is an "exact category". Our goal is to recover the highest weight category $\mathcal{C}_A$ (i.e. the triple $(\mathcal{C}_A, (\mathcal{T}, \leq), \Delta_A(\tau)_{\tau \in \mathcal{T}})$) from the exact category $\mathcal{C}_A^\Delta$.

1.1) Projective generator.

Set $P := \bigoplus_{\tau \in \mathcal{T}} P_\tau$ & $\tilde{E} = End_{\mathcal{C}_A}(P)^{opp}$

Lemma: 1) $\mathcal{F} := Hom_{\mathcal{C}_A}(P, \cdot) : \mathcal{C}_A \xrightarrow{\sim} \tilde{E}\text{-mod}_{fg}$

2) $\tilde{E}$ is free over A.

Proof:

1) By (HW1) & (HW4) $\forall M \in \mathcal{C}_A \setminus \{0\}$ admits a nonzero homomorphism from P. Moreover, if $M_1 \subsetneq M_2$ are subobjects of M, then $\mathcal{F}(M_1) \subsetneq \mathcal{F}(M_2)$ are subobjects in $\mathcal{F}(M)$.

Since $\mathcal{F}(M)$ is fin. gen'd, hence Noetherian A-module, M is a Noetherian object in $\mathcal{C}_A$. The rest of the proof repeats that in Sec 2.1 of Lec 8 & is left as an exercise.

2) Since all Δ's are flat over A, so is P_τ, so is $\mathcal{F}(P_\tau)$ ($\mathcal{F}$ is exact). Hence $\tilde{E}$ is flat. Since it's fin. generated, it's projective. And since A is local, $\tilde{E}$ is free. □

1.2) Simple objects

Let $\mathfrak{m} \subset A$ be the max. ideal & $\mathbb{K} := A/\mathfrak{m}$. Let $\mathcal{C}_{\mathbb{K}}$ denote the full subcategory $\{M \in \mathcal{C}_A \mid \mathfrak{m}M = 0\}$

Proposition: Every $\Delta_A(\tau)$ has a unique irreducible quotient & it lies in $\mathcal{C}_{\mathbb{K}}$, denote it by $L_{\mathbb{K}}(\tau)$. Move-

over, the map $\tau \mapsto L_{\mathbb{K}}(\tau)$ is a bijection $\mathcal{T} \xrightarrow{\sim} \mathrm{Irr}(\mathcal{C}_{\mathbb{K}})$.

Proof:

$\forall$ simple is a quotient of $\Delta_A(\tau)$ for some τ by (HW1) If $L \in \mathrm{Irr}(\tilde{E}) \Rightarrow \mathbb{K}L \neq L \Rightarrow \mathbb{K}L = 0$. The same holds for simples in $\mathcal{C}_A$. It remains to establish two claims:

i) If $L_1, L_2 \in \mathrm{Irr}(\mathcal{C}_{\mathbb{K}}) \Rightarrow \Delta_A(\tau)$ does not admit an epimorphism onto $L_1 \oplus L_2$

ii) If $\Delta_A(\tau), \Delta_A(\tau') \twoheadrightarrow L$ for $L \in \mathrm{Irr}(\mathcal{C}_{\mathbb{K}}) \Rightarrow \tau = \tau'$

Proof of i): $P_\tau \twoheadrightarrow \Delta(\tau) \twoheadrightarrow L_i$ $\forall$ $i = 1,2 \Rightarrow$ the $\mathbb{K}$-vector space $M_1 := \mathrm{Hom}_{\mathcal{C}_A}(P_\tau, L_1 \oplus L_2)$ has $\dim \geq 2$

On the other hand, by (HW4) & (HW2), $\ker[P_\tau \twoheadrightarrow \Delta_A(\tau)]$ has only zero homs to $\Delta_A(\tau)$, so (HW3) $\Rightarrow \mathrm{Hom}_{\mathcal{C}_A}(P_\tau, \Delta_A(\tau))$ $\mathrm{End}_{\mathcal{C}_A}(\Delta_A(\tau))$ [(HW3)] $\xrightarrow{\sim} A$. Since P_τ is projective, we have A-linear $A \twoheadrightarrow M_1$. This contradicts A being local.

Proof of ii) $P_\tau \twoheadrightarrow \Delta_A(\tau) \twoheadrightarrow L \twoheadleftarrow \Delta_A(\tau') \Rightarrow$

$\mathrm{Hom}_{\mathcal{C}_A}(P_\tau, \Delta_A(\tau')) \neq 0$ [HW4] $\Rightarrow \exists\, \tau_1 \geq \tau \mid \mathrm{Hom}_{\mathcal{C}_A}(\Delta_A(\tau_1), \Delta_A(\tau')) \neq 0$

$\Rightarrow$ [HW2] $\tau_1 \leq \tau' \Rightarrow \tau \leq \tau'$. Similarly, $\tau' \leq \tau \Rightarrow \tau' = \tau$. $\square$

We will use Proposition to describe $\Delta_A(\tau)$ for maximal $\tau \in \mathcal{T}$. By (HW4), $P_\tau \xrightarrow{\sim} \Delta_A(\tau)$. Now recall (Sec 2.3 of Lec 8), that one has primitive idempotents $\varepsilon_{\tau'} \in \tilde{E}$ s.t. $\dim_{\Bbbk} \varepsilon_{\tau'} L_{\Bbbk}(\tau'') = \delta_{\tau',\tau''}$. Moreover, a projective object P in $\tilde{E}\text{-mod}_{fg}$ is isomorphic to $\bigoplus_{\tau \in \mathcal{T}} (\tilde{E}\varepsilon_\tau)^{\oplus n_\tau}$ w. $n_\tau = \dim_{\Bbbk}(P, L_{\Bbbk}(t))$. Proposition implies:

Corollary. Under $\mathcal{C}_A \xrightarrow{\sim} \tilde{E}\text{-mod}_{fg}$, $\Delta_A(\tau) \mapsto \tilde{E}\varepsilon_\tau$.

1.3) Highest weight subcategories

Let $\underline{\mathcal{T}} \subset \mathcal{T}$ be a poset ideal (i.e. $\tau \leq \tau'$ & $\tau' \in \underline{\mathcal{T}} \Rightarrow \tau \in \underline{\mathcal{T}}$)

Definition: Define the full subcategory $\underline{\mathcal{C}}_A (= \mathcal{C}_A(\underline{\mathcal{T}}))$ as $\{M \in \mathcal{C}_A \mid \operatorname{Hom}_{\mathcal{C}_A}(\Delta_A(\tau'), M) = 0 \ \forall\, \tau' \notin \underline{\mathcal{T}}\}$

Example: Let $\mathcal{C}_A = \mathcal{O}_A^{\Xi}$ & $\mathcal{T} = \Xi$ (cf. Sec 2.2 in Lec 17)

Then $M \in \underline{\mathcal{C}}_A \Leftrightarrow \forall$ weight μ of M $\exists\, \tau \in \underline{\mathcal{T}}$ w. $\mu \leq \tau$

(exercise).

Exercise 1: 1) $M \in \underline{\mathcal{C}}_A \Leftrightarrow \mathrm{Hom}_{\mathcal{C}_A}(P_{\tau'}, M) = 0 \ \forall \tau' \notin \underline{T}$.

Deduce that $\underline{\mathcal{C}}_A$ is a Serre subcategory (closed under subs, quotients & extensions)

2) $\mathrm{Ext}^i_{\mathcal{C}_A}(\Delta(\tau'), M) = 0 \ \forall i > 0$, $M \in \underline{\mathcal{C}}_A$, $\tau' \notin \underline{T}$ (hint: use $P_\tau \to \Delta_A(\tau)$ & induction).

3) $\Delta_A(\tau) \in \underline{\mathcal{C}}_A \ \forall \tau \in \underline{T}$.

We'll need the following lemme

Lemme: Let $M \in \mathcal{C}_A^\Delta$.

1) There's a unique subobject $\overline{M} \subset M \mid \underline{M} := M/\overline{M}$ is filtered by $\Delta_A(\tau)$'s w. $\tau \in \underline{T}$ & $\overline{M}$ is filtered by $\Delta_A(\tau')$'s w. $\tau' \notin \underline{T}$.

2) Moreover $M \mapsto \overline{M}$ is functorial: if $\varphi \in \mathrm{Hom}_{\mathcal{C}_A}(M, N)$ & $N \in \mathcal{C}_A^\Delta \Rightarrow \varphi(\overline{M}) \subset \overline{N}$.

3) If $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact w. $M_i \in \mathcal{O}_A^\Delta$, then the sequences $0 \to \underline{M}_1 \to \underline{M}_2 \to \underline{M}_3 \to 0$, $0 \to \bar{M}_1 \to \bar{M}_2 \to \bar{M}_3 \to 0$ are exact as well.

Proof:

1): By 2) & 3) of Exercise 1, $\mathrm{Ext}^1_{\mathcal{O}_A}(\Delta_A(\tau'), \Delta_A(\tau)) = 0$ $\forall \tau \in \underline{J}$, $\tau \notin \underline{J}$. This gives the existence of $\bar{M} \subset M$.

It follows from (HW2) that $\mathrm{Hom}_{\mathcal{O}_A}(\bar{M}, \underline{N}) = 0$. This implies both the uniqueness & 2).

3): Suppose $M_\bullet = (0 \to M_k \to \dots \to M_0 \to 0)$ is a complex of objects filtered by $\Delta_A(\tau')$ w. $\tau' \notin \underline{J}$. Let $i = \min\{j \mid H_j(M_\bullet) = 0\}$. For $N \in \underline{\mathcal{O}}_A$,

$\mathrm{Hom}_{\mathcal{O}_A}(H_i(M_\bullet), N) = \mathrm{Hom}_{D^b(\mathcal{O}_A)}(M_\bullet, N[i]) = [3) \text{ of Exercise } 1] = 0$

Getting back to 3) consider morphisms of complexes $0 \to \bar{M}_\bullet \to M_\bullet \to \underline{M}_\bullet \to 0$. Since $M_\bullet$ is exact, from the long exact sequence in cohomology we get $H_i(\underline{M}_\bullet) \xrightarrow{\sim} H_{i-1}(\bar{M}_\bullet)$ $\forall i$. The claim above in the proof implies $H_i(\underline{M}_\bullet) \simeq H_{i-1}(\bar{M}_\bullet) = 0$ $\forall i$ □

Proposition: $\underline{\mathcal{C}}_A$ is a highest weight category w. poset $\underline{\mathcal{T}}$ & standards $\Delta_A(\tau)$, $\tau \in \underline{\mathcal{T}}$.

Proof: (HW0)-(HW3) for $\underline{\mathcal{C}}_A$ follow immediately from these properties for $\mathcal{C}_A$. Let's check (HW4), namely that $\underline{P}_\tau$ satisfies (HW4). Since $\mathrm{Hom}_{\mathcal{C}_A}(\bar{P}_\tau, M) = 0$ (1) of Exercise 1) we get $\mathrm{Hom}_{\mathcal{C}_A}(P_\tau, M) \xrightarrow{\sim} \mathrm{Hom}_{\underline{\mathcal{C}}_A}(\underline{P}_\tau, M)$ $\forall M \in \underline{\mathcal{C}}_A$. So $\underline{P}_\tau$ is projective in $\underline{\mathcal{C}}_A$. Let $K = \ker[P_\tau \to M]$, then $\underline{K} = \ker[\underline{P}_\tau \to M]$ & $\bar{K} = \bar{P}_\tau$ by 3) of Lemma. Since K admits a filtration by $\Delta_A(\tau')$ w. $\tau' > \tau$, $\underline{K}$ is filtered by those of them w. $\tau' \in \underline{\mathcal{T}}$ finishing the proof $\square$

1.4) Full faithfulness of Ext's.

Lemma: in the setting of Sec 1.3, $\forall M \in \underline{\mathcal{C}}_A^{\Delta}$, $N \in \underline{\mathcal{C}}_A$, $i \geq 0$, the map

$$\mathrm{Ext}^i_{\underline{\mathcal{C}}_A}(M, N) \to \mathrm{Ext}^i_{\mathcal{C}_A}(M, N)$$

induced by $\underline{\mathcal{C}}_A \hookrightarrow \mathcal{C}_A$ is an isomorphism.

Proof:

Pick $\{0\}=M_0\subset M_1\subset\ldots\subset M_r=M$ w. $M_i/M_{i-1}\simeq\Delta_A(\tau_i)$. Then we get an epimorphism $P_{\tau_i}\twoheadrightarrow M_i/M_{i-1}$ w. kernel, $K_i\in\mathcal{C}_A^\Delta$. This gives an epimorphism $P^0:=\bigoplus_{i=1}^r P_{\tau_i}\to M$ w. kernel, K^0, filtered by K_i's hence in $\mathcal{C}_A^\Delta$. We repeat w. K^0 getting a SES $0\to K^{-1}\to P^{-1}\to K^0\to 0$ etc. In particular, we get a projective resolution $\ldots\to P^{-2}\to P^{-1}\to P^0$ of M in $\mathcal{C}_A$ & (thx to Lemma in Sec 1.3) $\ldots\to\underline{P}^{-2}\to\underline{P}^{-1}\to\underline{P}^0$ of M in $\underline{\mathcal{C}}_A$. Since $\mathrm{Hom}_{\mathcal{C}_A}(P^i,M)=\mathrm{Hom}_{\underline{\mathcal{C}}_A}(\underline{P}^i,M)$ we are done. □

Premium exercise: Show that if A is regular, then $\mathrm{Ext}^i_{\underline{\mathcal{C}}_A}(M,N)\xrightarrow{\sim}\mathrm{Ext}^i_{\mathcal{C}_A}(M,N)$ $\forall\, M,N\in\underline{\mathcal{C}}_A$. Hint: start w. the case when $A=\mathbb{K}$.