$\mathcal{O}$ & Soergel, lec 1: Overview I

1) Category $\mathcal{O}$
2) Three theorems of Soergel

1) Category $\mathcal{O}$

1.1) Def'ns: $\mathbb{K}$ char 0 field

$\mathfrak{g}$ s/simple Lie alg $/\mathbb{K}$ $\rightsquigarrow$ univ. enveloping $\mathcal{U}(\mathfrak{g})$

$\mathfrak{g} \supset \mathfrak{b} = \mathfrak{h} \ltimes \mathfrak{n}$

Borel | $\Lambda \subset \mathfrak{h}^*$ weight lattice.

Def'n: Fix $\chi \in \mathfrak{h}^* \rightsquigarrow$ cat. $\mathcal{O}_\chi$ = full subcat in $\mathcal{U}(\mathfrak{g})$-Mod of all $M \in \mathcal{U}(\mathfrak{g})$-Mod s.t.

(i) M is fin. gen'd $/\mathcal{U}(\mathfrak{g})$

(ii) (weight decomp.) $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$

$M_\lambda = \{m \in M \mid xm = \langle \lambda + \chi, x \rangle m \ \forall x \in \mathfrak{h}\}$

(iii) $\mathfrak{n}$ acts on M locally nilpotently:

$\forall m \in M \ \exists r > 0 \mid y_1 \dots y_r m = 0 \ \forall y_1, \dots y_r \in \mathfrak{n}$.

Exer 0: $\mathcal{O}_\chi$ is an abelian subcat. in $\mathcal{U}(\mathfrak{g})$-Mod
closed under (co)kernels

Exer 1: Modulo (i) & (ii), (iii) $\Leftrightarrow$
(iii'): $\{\lambda\in\Lambda \mid M_\lambda \neq 0\}$ is "bounded from above":
$\exists\, \lambda_1, \dots \lambda_k \in \Lambda$ s.t. $M_\lambda \neq 0 \Rightarrow \exists\, i$ s.t. $\lambda \leq \lambda_i$
$\lambda_i - \lambda$ = sum of positive roots.

Exer 2: $\dim M_\lambda < \infty\ \forall\, \lambda\in\Lambda$.

Examples of objects:
1) Verma modules $\Delta_\chi(\lambda)$ $(\lambda\in\Lambda)$
$\Delta_\chi(\lambda) = \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})\,\mathrm{Span}_{\mathbb{C}}(x - \langle\lambda+\chi, x\rangle, y \mid x\in\mathfrak{h}, y\in\mathfrak{n})$.

2) $\forall\, \lambda\in\Lambda$, $\Delta_\chi(\lambda)$ has unique irred. quotient, denoted by $L_\chi(\lambda)$; $\lambda \mapsto L_\chi(\lambda): \Lambda \xrightarrow{\sim} \mathrm{Irr}(\mathcal{O}_\chi)$
$\{$irred. objects in $\mathcal{O}_\chi\}$

proofs: exercises.

Exercise 3 (univ. property of $\Delta_\chi(\lambda)$)

$\mathrm{Hom}_{\mathcal{O}_\chi}(\Delta_\chi(\lambda), M) \xrightarrow{\sim} \{m \in M_\lambda \mid \mathfrak{n}m = 0\}$

Rem: $\chi - \chi' \in \Lambda \Rightarrow \mathcal{O}_\chi \xrightarrow{\sim} \mathcal{O}_{\chi'}$, $M \mapsto M$ (shifts weight decomposition). The most interesting is $\chi = 0$.

1.2) Structural features

$W :=$ Weyl group, $\rho = \frac{1}{2}\sum$ positive roots

$Z =$ center of $\mathcal{U}(\mathfrak{g})$

Harish-Chandra: $Z \xrightarrow{\sim} \mathbb{K}[\mathfrak{h}^*]^{(W,\cdot)}$, $z \mapsto HC_z$

$w \cdot \lambda = w(\lambda+\rho) - \rho$

z acts on $\Delta_\chi(\lambda)$ by $HC_z(\lambda + \chi)$.

"Infinitesimal block decomposition":

Thm 1: $\mathcal{O}_\chi = \bigoplus_{\xi \in \underbrace{\mathfrak{h}^*/(W,\cdot)}_{\text{set of orbits}}} \mathcal{O}_\chi^\xi$

$\mathcal{O}_\chi^\xi = \{M \in \mathcal{O}_\chi \mid \forall m \in M \ \exists n > 0 \text{ s.t. } (z - HC_z(\xi))^n m = 0 \ \forall z \in Z\}$

Exercise: 1) $\Delta_\chi(\lambda), L_\chi(\lambda) \in \mathcal{O}_\chi^\xi \Leftrightarrow \xi = W\cdot(\lambda+\chi)$

2) $\Leftarrow$ If $\mathcal{O}_\chi^\xi \neq 0 \Rightarrow \exists\ \lambda \in \Lambda \mid \xi = W\cdot(\lambda+\chi)$.

Thm 2: Each $\mathcal{O}_\chi^\xi$

(i) has fin. many simple objects.

(ii) $\forall\ M, N \in \mathcal{O}_\chi^\xi \Rightarrow \dim \operatorname{Hom}_{\mathcal{O}_\chi^\xi}(M,N) < \infty$

(iii) has enough projectives: $\forall\ M \in \mathcal{O}_\chi^\xi\ \exists$ projective $P_M \mid P_M \twoheadrightarrow M$.

Application of Thm 2:

$L = \bigoplus$ all simple objects $\in \mathcal{O}_\chi^\xi$

$\rightsquigarrow$ [(iii)] projective $P_L \in \mathcal{O}_\chi^\xi \mid P_L \twoheadrightarrow L$.

$A := \operatorname{End}_{\mathcal{O}_\chi^\xi}(P_L)$ – fin. dim. assoc. algebra

by (ii)

$\operatorname{Hom}_{\mathcal{O}_\chi^\xi}(P_L, ?) : \mathcal{O}_\chi^\xi \xrightarrow{\sim} \operatorname{mod}_{fd}\text{-}A$

fin. dim. right A-modules.

Preliminary conclusion: $\mathcal{O}_\chi^\zeta$ can be recovered from A, equivalently, from its additive subcat. $\mathcal{O}_\chi^\zeta$-proj of the projective objects.

Soergel: gives elementary ($\neq$ easy) description of $\mathcal{O}_\chi^\zeta$-proj.

2) Three theorems of Soergel

2.0) Digression: indecomposable objects

$\mathcal{C}$ $\mathbb{K}$-linear abelian category

Say, $M \in \mathcal{C}$ is indecomposable if $M \simeq M_1 \oplus M_2$ $\Rightarrow M_1 = 0$ or $M_2 = 0$.

Assume

(*) $\forall\, M \in \mathcal{C} \Rightarrow \dim \operatorname{End}_{\mathcal{C}}(M) < \infty$.

One can read indecomposability of M from $\operatorname{End}_{\mathcal{C}}(M)$ (as algebra): if $\varepsilon \in \operatorname{End}_{\mathcal{C}}(M)$ is idempotent ($\varepsilon^2 = \varepsilon$), $\varepsilon \neq 0, 1 \Rightarrow M = \operatorname{im} \varepsilon \oplus \operatorname{im}(1-\varepsilon)$ - nontrivial.

Exercise: Assume (*).

1) TFAE a) M is indecomposable

b) $\mathrm{End}_{\mathcal{C}}(M)/\mathrm{rad}\,\mathrm{End}_{\mathcal{C}}(M)$ is division alg.

2) $M \simeq \bigoplus$ of indecomposables

Thm (Krull-Schmidt) Assume (*). Then decomposition in 2) is unique: if $M = M_1 \oplus \ldots \oplus M_k = M_1' \oplus \ldots \oplus M_\ell' \Rightarrow$ $k=\ell \Rightarrow \exists\, \sigma \in S_k \mid M_i \simeq M'_{\sigma(i)}$.

2.1) First two theorems.

For simplicity: $\chi = 0$, $\xi = W\cdot 0$, $\mathcal{O}^{pr} := \mathcal{O}_0^{W\cdot 0}$ (principal block). The simple objects: $L_0(w\cdot 0)$ ($w\cdot 0 = w\rho - \rho$ are pairwise distinct; $w\in W$)

Rep. theory of fin. dim. assoc. algebras:

$\forall\, w\in W\ \exists!$ <u>indecomposable</u> projective surjecting onto $L_0(w\cdot 0)$, notation $P_0(w\cdot 0)$ ($\Rightarrow \mathrm{Hom}_{\mathcal{O}^{pr}}(P_0(w\cdot 0), L_0(u\cdot 0))$ has dim $\delta_{w,u}$).

Want to describe $\mathrm{End}_{\mathcal{O}^{pr}}(\bigoplus_{w\in W} P_0(w\cdot 0))$

1st theorem: computation $\mathrm{End}_{\mathcal{O}^{pr}}(P_0(w_0\cdot 0))$, where $w_0 \in W$ is the longest element (w_0 sends dominant Weyl chamber, C, to $-C \Rightarrow w_0 \rho = -\rho \Rightarrow w_0 \cdot \rho = -2\rho$.

$\mathbb{K}[\mathfrak{h}^*]^W \supset \mathbb{K}[\mathfrak{h}^*]^W_+ = \{f \mid f(0)=0\}$ - max. ideal

usual action

"Coinvariant algebra":

$\mathbb{K}[\mathfrak{h}^*]^{coW} = \mathbb{K}[\mathfrak{h}^*] / \mathbb{K}[\mathfrak{h}^*]\,\mathbb{K}[\mathfrak{h}^*]^W_+$

- fin. dim., commut., $\dim = |W|$

Thm 1 (Soergel) $\mathrm{End}_{\mathcal{O}^{pr}}(P_0(w_0\cdot 0)) \xrightarrow{\sim} \mathbb{K}[\mathfrak{h}^*]^{coW}$.

as assoc. algebras.

Functor $\mathbb{V} := \mathrm{Hom}_{\mathcal{O}^{pr}}(P_0(w_0\cdot 0), ?) : \mathcal{O}^{pr} \longrightarrow \mathbb{K}[\mathfrak{h}^*]^{coW}\text{-mod}_{fd}$

NOT an equivalence: $L(w\cdot 0) \mapsto 0$ if $w \neq w_0$

Thm 2 (Soergel) $\mathbb{V}$ is fully faithful on projectives:

$\mathrm{Hom}_{\mathcal{O}^{pr}}(P_1, P_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{K}[\mathfrak{h}^*]^{coW}}(\mathbb{V}(P_1), \mathbb{V}(P_2))$

$\forall\, P_1, P_2 \in \mathcal{O}^{pr}\text{-proj.}$

2.2) Third theorem

$\mathcal{O}^{pr}$-proj $\leftsquigarrow$ compute $\mathbb{V}(P_0(w\cdot 0))$ $\forall\, w\in W$.

by Thm 2: indecomposable $\mathbb{K}[\mathfrak{h}^*]^{coW}$

Two features of $\mathcal{O}^{pr}$:

1) Verma $\Delta_0(0)$ is projective; $\mathbb{V}(\Delta_0(0))\simeq \mathbb{K}$

$$\mathbb{K} \cong \mathbb{K}[\mathfrak{h}^*]/(\mathfrak{h}^*)$$

2) $\forall$ simple reflection $s\in W$ have "reflection functor" $\Theta_s : \mathcal{O}^{pr}\longrightarrow \mathcal{O}^{pr}$. We know:

i) Θ_s is self-biadjoint $\Rightarrow$ exact, preserves $\mathcal{O}^{pr}$-proj

ii) If $\ell(ws)>\ell(w)$ ($\ell: W\to \mathbb{Z}_{\geqslant 0}$ length function) $P_0(ws\cdot 0)$ occurs as $\oplus$-summand in $\Theta_s P(w\cdot 0)$.

iii) $w=s_1\ldots s_\ell$ reduced decomp'n $\Rightarrow$ $P_0(w\cdot 0)$ occurs w. mult. 1 in $\Theta_{s_\ell}\ldots\Theta_{s_1}\Delta(0)$ & all other $P_0(u\cdot 0)$ that occur satisfy $\ell(u)<\ell(w)$.

Endofunctor of $\mathbb{K}[\mathfrak{h}^*]^{coW}$-mod:

$\mathbb{K}[\mathfrak{h}^*]\otimes_{\mathbb{K}[\mathfrak{h}^*]^s}\bullet : \mathbb{K}[\mathfrak{h}^*]\text{-mod} \circlearrowleft$

$\mathbb{K}[\mathfrak{h}^*]^s = \{f\in\mathbb{K}[\mathfrak{h}^*] \mid sf=f\}$

Restricts to $\mathbb{K}[\mathfrak{h}^*]^{coW}$-mod $\subset \mathbb{K}[\mathfrak{h}^*]$-mod
(exercise: $\mathbb{K}[\mathfrak{h}^*]^W \subset \mathbb{K}[\mathfrak{h}^*]^s$)

Thm 3: $\forall$ simple reflection $s \in W$ $\exists$ functor iso
$\mathbb{V} \circ \Theta_s \xrightarrow{\sim} \mathbb{K}[\mathfrak{h}^*] \otimes_{\mathbb{K}[\mathfrak{h}^*]^s} \mathbb{V}$

Summary: $\mathcal{O}^{pr}$-proj $\xrightarrow{\sim}$ full subcat. in $\mathbb{K}[\mathfrak{h}^*]^{coW}$-mod consisting of direct sums of direct summands of modules $\mathbb{K}[\mathfrak{h}^*] \otimes_{\mathbb{K}[\mathfrak{h}^*]^{s_1}} \mathbb{K}[\mathfrak{h}^*] \otimes_{\mathbb{K}[\mathfrak{h}^*]^{s_2}} \otimes \ldots \otimes_{\mathbb{K}[\mathfrak{h}^*]^{s_k}} \mathbb{C}$
$s_1, \ldots s_k \in W$ are simple reflections.