$\mathcal{O}$ & Soergel, lec 2: Overview II

1) Soergel (bi)modules

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1.1) Definition & examples

Let W be a finite Coxeter group & $\mathfrak{h}$ its reflection representation. In Sec 2 of Lec 1, in the case when W is the Weyl group of $\mathfrak{g}$ we've embedded $\mathcal{O}^{pr}$-proj (where $\mathcal{O}^{pr}$ is the principal block of category $\mathcal{O}$ for $\mathfrak{g}$) into $\mathbb{K}[\mathfrak{h}^*]^{coW}$-mod and gave a description of the image.

Now we elaborate on modules in the image.

The algebra $R:=\mathbb{K}[\mathfrak{h}^*]=S(\mathfrak{h})$ is graded w. $\deg \mathfrak{h}=2$. We have the abelian category R-grbimod of $\mathbb{Z}$-graded R-bimodules. It is monoidal w.r.t. the tensor product of bimodules, $\otimes_R$, that we'll omit in the notation: $B_1B_2:=B_1\otimes_R B_2$; the unit is R w. its usual grading. And R-grbimod comes w. the grading shift functor

 $\langle 1\rangle$: if $B=\bigoplus_i B_i$, then $B\langle 1\rangle = B$ w. $B\langle 1\rangle_i := B_{i+1}$

Let $S \subset W$ be the subset of all simple reflections.

Def'n: 1) $s \in S \rightsquigarrow$ elementary Bott-Samelson bimodule $BS_s := R \otimes_{R^s} R\langle 1 \rangle$ (so that $\deg 1\otimes 1 = -1$). For $\underline{w} := (s_1, \dots, s_k) \in S^k$ define the general Bott-Samelson bimodule by

$$BS_{\underline{w}} = BS_{s_1} BS_{s_2} \dots BS_{s_k} = R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \dots \otimes_{R^{s_k}} R\langle k \rangle$$

2) The category $SBim(W)$ of Soergel bimodules is the minimal additive monoidal subcategory of R-grbimod containing BS_s $\forall s \in S$ & closed under grading shifts $\langle \pm 1 \rangle$ & taking direct summands

3) The category $SMod(W)$ of Soergel modules is the full (additive) subcategory of R-grmod whose objects are direct summands of $B \otimes_R \mathbb{K}_0$ for $B \in SBim(W)$

Exercise 1:

1) $BS_{(s_1, \dots s_k)}$ is a free rank 2^k graded right R-module

2) $\forall$ $B \in SBim(W)$ is a free graded right R-module.

Remarks 1) In SBim & SMod the decomposition of any object into indecomposable direct summands is unique by the Krull-Schmidt thm. Indeed Hom's are finite dimensional (exercise).

2) On any $BS_{\underline{w}}$ the actions of R^W from the left & from the right coincide (exercise). In particular, $\forall$ object from SBim is a fin. generated module over $R\otimes_{R^W} R$ & $\forall$ object of SMod is a fin. dim. module over R^{coW}.

Example 1: $W=S_2$, $\mathfrak{h}=\mathbb{K}$, $R=\mathbb{K}[\overset{\text{deg }2}{x}]$, $sx=-x$, $R^s=\mathbb{K}[x^2]$

$BS_{(s,s)}=\mathbb{K}[x]\otimes_{\mathbb{K}[x^2]}\mathbb{K}[x]\otimes_{\mathbb{K}[x^2]}\mathbb{K}[x]\langle 2\rangle = [\mathbb{K}[x]=\mathbb{K}[x^2]\oplus \mathbb{K}[x^2]\langle -2\rangle$ as graded $\mathbb{K}[x^2]$-bimodule$]$

$=\mathbb{K}[x]\otimes_{\mathbb{K}[x^2]}\mathbb{K}[x]\langle 2\rangle \oplus \mathbb{K}[x]\otimes_{\mathbb{K}[x^2]}\mathbb{K}[x] = BS_s\langle 1\rangle \oplus BS_s\langle -1\rangle.$

So the objects of $SBim(S_2)$ are direct sums of graded shifts of BS_s & $R=\mathbb{K}[x]$ (exercise).

3] The following will be proved later in the course.

Proposition: For $B \in SBim$ TFAE:

1) B is indecomposable in R-grbimod ($\Leftrightarrow$ in $SBim$)

2) $B \otimes_R \mathbb{K}_0$ is indecomposable in R-grmod ($\Leftrightarrow$ in $SMod$)

3) $B \otimes_R \mathbb{K}_0$ is indecomposable in R-mod

Moreover, if $B_1, B_2 \in SBim$ are indecomposable & $B_1 \otimes_R \mathbb{K}_0 \xrightarrow[R]{\sim} B_2 \otimes_R \mathbb{K}_0$, then $B_1 \simeq B_2\langle i \rangle$ in R-grbimod for unique i. This gives a bijection between indecomposables in $SBim$ & $SMod$.

Example 2: Let $W = S_3$. We claim that (up grading shifts) there are 6 indecomposable Soergel bimodules. Let $S = \{s, t\}$ ($s = (12)$ $t = (23) \in S_3$). Then the indecomposables are

(1) $R, BS_s, BS_t, BS_{(s,t)}, BS_{(t,s)}, R \otimes_{R^W} R\langle 3 \rangle$

Here's how one approaches this

i) One can prove by hand that $BS_{(s,t)} \otimes_R \mathbb{C}_0$ is indecomposable. So by an easy part of Proposition above, $BS_{(s,t)}$ is indecomposable. Similarly, $BS_{(t,s)}$ is indecomposable.

ii) $\cdot\ BS_{(s,t,s)} \simeq R \otimes_{R^W} R\langle 3\rangle \oplus BS_s$

$\cdot\ BS_{(t,s,t)} \simeq R \otimes_{R^W} R\langle 3\rangle \oplus BS_t$

One can see this in an elementary way or deduce this from geometry.

Exercise: Use i) & ii) to show that bimodules in 1) exhaust indecomposable objects in SBim (up to shifts).

1.2) Application to category $\mathcal{O}$

Suppose now W is the Weyl group of $\mathfrak{g}$.

Let $SMod_{ugr}(W)$ ("ugr" - ungraded) denote the category whose objects are Soergel modules & morphisms are all (not necessarily graded) R-linear maps.

Soergel's Thms (Sec 2 of Lecture 1) give a full embedding $\mathbb{V}$ from the category $\mathcal{O}^{pr}$-proj of projective objects in $\mathcal{O}^{pr}$ to R-mod (Thms 1 & 2) & show

$$\mathbb{V}(\Theta_{s_1}\dots\Theta_{s_\ell}\Delta_0(0)) \simeq BS_{(s_1,\dots s_\ell)} \otimes_R \mathbb{K}_0 \quad (\text{Thm } 3)$$

The projectives $\Theta_{s_1}\ldots\Theta_{s_\ell}\Delta(0)$ contain every indecomposable projective $P_0(w\cdot 0)$ as a direct summand. For general reasons (in the representation theory of fin. dim. associative algebras) every object in $\mathcal{O}^{pr}$-proj (uniquely) decomposes as the direct sum of $P_0(w\cdot 0)$'s.

Thanks to 2) $\Leftrightarrow$ 3) in Proposition (in Sec 1.1) $\forall$ indecomposable in SMod remains indecomposable in $SMod_{ugr}$. So $SMod_{ugr}$ consists of direct sums of direct R-module summands of $BS_{(s_1,\ldots s_\ell)}\otimes_R \mathbb{K}_0$. Hence

(1) $\quad \mathbb{V}: \mathcal{O}^{pr}\text{-proj} \xrightarrow{\sim} SMod_{ugr}(W)$

equivalence of additive categories.

We would like to deduce a representation theoretic corollary. Consider the Langlands dual Lie algebra $\check{\mathfrak{g}}$ defined so that the roots for $\check{\mathfrak{g}}$ are the coroots for $\mathfrak{g}$. Hence for simple $\mathfrak{g}$ we have $\mathfrak{g}\simeq\check{\mathfrak{g}}$ unless $\mathfrak{g}\simeq\mathfrak{so}_{2n+1}$ or $\mathfrak{sp}_{2n}$ for $n\geqslant 3$, these are swapped). In particular, the Weyl groups of $\mathfrak{g}$ & $\check{\mathfrak{g}}$ are the same & $\check{\mathfrak{h}} \xrightarrow{\sim} \mathfrak{h}^*$ $\xrightarrow{\sim} \mathfrak{h}$ where the first isomorphism is canonical & the 2nd is any W-equivariant isomorphism

$SMod(W)$ depends only on W & its reflection repres'n.

So $\mathcal{O}^{pr}(\mathfrak{g})\text{-proj} \xrightarrow[\nabla]{\sim} SMod_{ugr}(W) \xleftarrow[\check{\nabla}]{\sim} \mathcal{O}^{pr}(\check{\mathfrak{g}})\text{-proj}$

Corollary: $\mathcal{O}^{pr}(\mathfrak{g}) \xrightarrow{\sim} \mathcal{O}^{pr}(\check{\mathfrak{g}})$ as abelian categories.

1.3) Soergel's categorification theorem

As we mentioned in Sec 2.2 of Lec 1, if $w = s_1 \dots s_\ell$ is a reduced expression, then

(2) $\Theta_{s_\ell} \dots \Theta_{s_1} \Delta(0) \simeq P_0(w\cdot 0) \oplus \bigoplus_{u \mid \ell(u)<\ell(w)} P_0(u\cdot 0)^{\oplus ?}$

Proposition in Sec 1.1 & (1) in Sec 1.2 yield:

(a) $\exists!$ indecomposable $B_w \in SBim$ w. (for $\underline{w} = (s_\ell, \dots s_1)$)

(3) $BS_{\underline{w}} \simeq B_w \oplus \bigoplus_{u \mid \ell(u)<\ell(w)} B_u\langle ?\rangle^{\oplus ?}$

(b) any indecomposable in $SBim$ is isomorphic to $B_w\langle i\rangle$ for uniquely determined $w \in W$, $i \in \mathbb{Z}$.

For example, $B_1 \simeq R$, $B_s = BS_s$ $\forall s \in S$.

How do $BS_{\underline{w}}$, $B_u B_w$ decompose into $\oplus$ of indecomposables?

This is answered (in a way) by the Soergel catego-rification theorem. It involves the Hecke algebra $\mathcal{H}(W)$ over $\mathbb{Z}[v^{\pm 1}]$: w. basis H_w $(w \in W)$ & multiplication uniquely recovered from $H_w H_s = \begin{cases} H_{ws} & \text{if } \ell(ws) = \ell(w)+1 \\ H_{ws} + (v^{-1}-v) H_w, & \text{else} \end{cases}$

On the other hand if we have an additive category $\mathcal{A}$ we can consider its split Grothendieck group $K_0(\mathcal{A})$. its generated by $[M]$ w. $M \in \mathcal{A}$ w. relations:

$$[M_1 \oplus M_2] = [M_1] \oplus [M_2]$$

If $\mathcal{A}$ is monoidal, $K_0(\mathcal{A})$ is an associative ring w.r.t. $[M_1][M_2] := [M_1 \otimes M_2]$. And if we have a grading shift functor $\langle 1 \rangle$, then $K_0(\mathcal{A})$ is a $\mathbb{Z}[v^{\pm 1}]$-module w. $v[M] = [M\langle 1 \rangle]$.

So $K_0(SBim)$ is a $\mathbb{Z}[v^{\pm 1}]$-algebra & free $\mathbb{Z}[v^{\pm 1}]$-module w. basis $[B_w]$.

Thm (Soergel) 1) $\exists !$ $\mathbb{Z}[v^{\pm 1}]$-algebra homomorphism

 $\mathcal{H}(W) \longrightarrow K_0(SBim)$ w. $H_s + v \mapsto [B_s]$. It's an isomorphism.

2) $[B_w] \mapsto$ "Kazhdan-Lusztig basis element c_w".

Sketch of homom'm part in 1) for simply laced cases:
$\mathcal{H}(W)$ is generated by H_s $(s\in S)$ subject to:

- braid relations: $H_sH_t = H_tH_s$ if $st=ts$ (4)
  $H_sH_tH_s = H_tH_sH_t$ if $sts=tst$ (5)
- quadratic relation $(H_s+v)(H_s-v^{-1})=0$ (6)

So we need to check analogs of (4)-(6) on elem'ts H_s+v for the elements $[B_s]$. This translates to

- $[B_s][B_t]=[B_t][B_s]$ for (4) (exercise: both are $BS_{(s,t)}$).
- $[B_s]^2=(v+v^{-1})[B_s]$ for (6) – from Example 1 in Sec 1.1
- $[B_s][B_t][B_s]-[B_s]=[B_t][B_s][B_t]-[B_t]$ for (5) – from decomposition of $BS_{(s,t,s)}$ in Example 2 in Sec. 1.1. $\square$

1) for non-simply laced types can be handled similarly but harder. The claim about isomorphism then follows from the classification of indecomposables in 6) above.

2) is a difficult part – Soergel's proof relied on geometry; we'll start discussing the relevant geometry in the next lecture.