O & Soergel, lec 2: Overview II

1) Soergel (bi)modules

1.1) Def'ns & examples

W is (finite) Coxeter group w. $W \curvearrowright \mathfrak{h}$ refl'n rep'n.

If W is Weyl group of $\mathfrak{g}$ (fin. dim & s/simple) $\rightsquigarrow$

$\mathcal{O}^{pr} \supset \mathcal{O}^{pr}\text{-proj} \xhookrightarrow[\mathbb{V}]{} \mathbb{K}[\mathfrak{h}^*]^{coW}\text{-mod}$ $(\mathbb{K}[\mathfrak{h}^*]^{coW} \twoheadleftarrow \mathbb{K}[\mathfrak{h}^*])$

modules in the image?

$R := \mathbb{K}[\mathfrak{h}^*] = S(\mathfrak{h})$ graded w. $\deg \mathfrak{h} = 2$

$\rightsquigarrow$ Abelian category $R\text{-grbimod}$ (objects = graded R-bimod's; morphisms = graded R-bilinear maps)

- monoidal w.r.t. $\otimes_R$; notation $B_1B_2 := B_1 \otimes_R B_2$
- unit : R w. usual grading
- grading shift functor $\langle 1 \rangle$: $B = \bigoplus_{i \in \mathbb{Z}} B_i$, $B\langle 1\rangle = B$

$B\langle 1 \rangle_i = B_{i+1}$.

$S \subset W$ simple reflections.

Def'ns: 1) $s \in S \rightsquigarrow$ elementary Bott-Samelson bimodule $BS_s := R \otimes_{\underbrace{R^s}_{s\text{-invariants}}} R\langle 1 \rangle$ ($\deg 1\otimes 1 = -1$)

$\underline{w} = (s_1, .. s_k) \in S^k \rightsquigarrow$ BS bimodule $BS_{\underline{w}} = BS_{s_1} BS_{s_2} \dots BS_{s_k}$
$= R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \dots \otimes_{R^{s_k}} R\langle k \rangle$.

2) Category $SBim(W)$ of Soergel bimodules = min'l full additive monoidal subcategory of R-grbimod s.t. $BS_s \in SBim(W)$ $\forall s \in S$, closed under $\langle \pm 1 \rangle$ & direct summands.

3) Category $SMod(W)$ of Soergel modules = full (additive) subcategory of R-grmod whose objects are (direct summands of) $B \otimes_R \mathbb{K}_0$ for $B \in SBim(W)$

Exercise 1: 1) $BS_{(s_1, .. s_k)}$ is free $rk = 2^k$ graded right R-module

2) $\forall B \in SBim(W)$ is free graded right R-module

Remarks: 1) In SBim & SMod decomp'n into indecomp's is unique by Krull-Schmidt theorem b/c Hom's are finite dimension (exercise).

2) On any $BS_{\underline{w}}$ ($\Rightarrow B \in SBim(W)$) left and right actions of R^W coincide (exercise) $\Rightarrow B \in R \otimes_{R^W} R$-gr bimod $\Rightarrow \forall$ Soergel module is a graded module$/R^{coW}$.

Example 1: $W = S_2 = \{\pm 1\} \curvearrowright \mathfrak{h} = \mathbb{K}$, $R = \mathbb{K}[\overset{\deg 2}{x}]$, $sx = -x$

$R^s = \mathbb{K}[x^2]$

$BS_{(s,s)} = \mathbb{K}[x] \otimes_{\mathbb{K}[x^2]} \mathbb{K}[x] \otimes_{\mathbb{K}[x^2]} \mathbb{K}[x]\langle 2\rangle =$

[as $\mathbb{K}[x^2]$-bimodule $\mathbb{K}[x] = \mathbb{K}[x^2] \oplus \mathbb{K}[x^2]\langle -2\rangle$]

$= \mathbb{K}[x] \otimes_{\mathbb{K}[x^2]} \mathbb{K}[x]\langle 2\rangle \oplus \mathbb{K}[x] \otimes_{\mathbb{K}[x^2]} \mathbb{K}[x]$

$= BS_s\langle -1\rangle \oplus BS_s\langle 1\rangle$.

Exercise: Objects in $SBim(S_2) = \oplus$'s of $BS_s\langle ?\rangle$ & $R\langle ?\rangle$.

Proposition: • $\forall$ $B \in SBim(W)$ TFAE:

1) B is indecomposable in R-grbimod

$\Uparrow$

2) $B \otimes_R \mathbb{K}_0$ — · — · — · — · in R-grmod easy

$\Uparrow$

3) — · — · — · — · — · — · in R-mod

• if $B_1, B_2 \in SBim$ are indecomposable s.t.

$B_1 \otimes_R \mathbb{K}_0 \xrightarrow{\sim} B_2 \otimes_R \mathbb{K}_0$ in R-mod $\Rightarrow \exists!\ i \in \mathbb{Z} \mid$

$B_1 \simeq B_2 \langle i \rangle$

$\Rightarrow$ indec.'s in $SBim \xrightarrow{\sim}$ indec. in $SMod$

Example 2 ($W = S_3$): up to shifts have 6 indec.

Soergel bimodules: $S = \{s, t\}$ $s = (12), t = (23) \in S_3$

Indecomposables: $R, BS_s, BS_t, BS_{(s,t)}, BS_{(t,s)}$,

$R \otimes_{R^W} R \langle 3 \rangle$.

Sketch:

i) Can prove $BS_{(s,t)} \otimes_R \mathbb{K}_0$ is indec. in R-grmod

$BS_{(t,s)} \otimes_R \mathbb{K}_0$ — · — · — · — · — · —

ii) $BS_{(s,t,s)} \simeq R \otimes_{R^W} R \langle 3 \rangle \oplus BS_s$

$BS_{(t,s,t)} \simeq R \otimes_{R^W} R \langle 3 \rangle \oplus BS_t$.

Exercise: Use i) & ii) to show that the 6 bimodules above exhaust all indecomposables in $SBim(S_3)$.

1.2) Applications to $\mathcal{O}$

W is Weyl group $\mathfrak{g}$

$SMod_{ugr}(W)$ "ugr" = ungraded

full subcategory of R-mod whose objects are Soergel modules.

Soergel's Thms 1 & 2: $\mathbb{V}: \mathcal{O}^{pr}\text{-proj} \hookrightarrow R\text{-mod}$

Thm 3 $\Rightarrow \mathbb{V}(\underline{\Theta_{s_\ell}\Theta_{s_{\ell-1}}\dots\Theta_{s_1}\Delta_0(0)}) \simeq BS_{(s_1,\dots s_\ell)}\otimes_R \mathbb{K}_0$

contain $\forall$ indecomposable $P_0(w\cdot 0)$

So: equiv. of additive cat's

(1) $\mathbb{V}: \mathcal{O}^{pr}\text{-proj} \xrightarrow{\sim} SMod_{ugr}(W)$.

$\mathfrak{g} \rightsquigarrow$ Langlands dual $\check{\mathfrak{g}}$ ($\check{\mathfrak{h}} = \mathfrak{h}^*$, roots of $\check{\mathfrak{g}}$ = coroots of $\mathfrak{g}$)

For simple $\mathfrak{g} \simeq \check{\mathfrak{g}}$ (if $\mathfrak{g}$ is not $B_n \leftrightarrow C_n$)

$\Rightarrow$ Weyl groups are the same

$\check{\mathfrak{h}} \xrightarrow{\sim} \mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}$ some W-invariant.

Hence: $\mathcal{O}^{pr}(\mathfrak{g})\text{-proj} \xrightarrow[\mathbb{V}]{\sim} SMod_{ugr}(W) \xleftarrow[\check{\mathbb{V}}]{\sim} \mathcal{O}^{pr}(\check{\mathfrak{g}})\text{-proj}$

Corollary: $\mathcal{O}^{pr}(\mathfrak{g}) \xrightarrow{\sim} \mathcal{O}^{pr}(\check{\mathfrak{g}})$ as abelian cats.

1.3) Soergel categorification thm

Lec 1, Sec 2.2: if $w = s_1 \dots s_\ell$ $(\ell = \ell(w)) \Rightarrow$

(2) $\Theta_{s_\ell} \dots \Theta_{s_1} \Delta_0(0) = P_0(w\cdot 0) \oplus \bigoplus_{u \mid \ell(u) < \ell(w)} P_0(u\cdot 0)^{\oplus ?}$

$\Big\downarrow$ (1)

indecomp's in $SMod_{ugr} \xleftrightarrow{\sim} W$

$\Big\downarrow$ Prop. in Sec 1.1

(a) $\exists!$ indecomposable $B_w \in SBim$ s.t. $(\underline{w} = (s_\ell, \dots s_1))$:

(3) $BS_{\underline{w}} \simeq B_w \oplus \bigoplus_{u \mid \ell(u) < \ell(w)} B_u \langle ? \rangle^{\oplus ?}$

(6) $\forall$ indec. in $SBim \simeq B_w\langle i \rangle$ for unique $w \in W, i \in \mathbb{Z}$.

Examples: $B_1 \simeq R$, $B_s = BS_s$.

Q: Decomposition of $BS_{\underline{w}}$ or $B_u B_w$?

To answer: need Hecke algebra $\mathcal{H}(W)/\mathbb{Z}[v^{\pm 1}]$

- has basis H_w $(w \in W)$
- multiplication: $H_w H_s = \begin{cases} H_{ws} & \text{if } \ell(ws) = \ell(w)+1 \\ H_{ws} + (v^{-1} - v) H_w, & \text{else} \end{cases}$

Additive category $\mathcal{A} \rightsquigarrow$ <u>split</u> Grothendieck group
$K_0(\mathcal{A}) = \langle [M] \mid M \in \mathcal{A} \rangle / ([M_1 \oplus M_2] = [M_1] \oplus [M_2])$.
abelian group.

If $\mathcal{A}$ is monoidal $\Rightarrow K_0(\mathcal{A})$ is assoc. unital ring $[M_1][M_2] = [M_1 \otimes M_2]$

If $\mathcal{A}$ has grading shift $\langle 1 \rangle$: $K_0(\mathcal{A})$ is $\mathbb{Z}[v^{\pm 1}]$-module $v[M] := [M\langle 1 \rangle]$

So $K_0(\mathrm{SBim})$ is assoc. unital $\mathbb{Z}[v^{\pm 1}]$-algebra

Krull-Schmidt + classif. of indec's:

$K_0(\mathrm{SBim})$ is free $\mathbb{Z}[v^{\pm 1}]$-module w. basis $[B_w]$.

Thm (Soergel):

1) $\exists!$ $\mathbb{Z}[v^{\pm 1}]$-algebra homom. $\mathcal{H}(W) \longrightarrow K_0(SBim)$ w. $H_s + v \mapsto [B_s]$. It's isom.

2) $[B_w] \mapsto$ "Kazhdan-Lusztig basis element c_w."

Proof of 1), homom. part: simply laced $\Leftarrow$ Ex 1&2 in Sec 1.1.

2): using geometry