$\mathcal{O}$ & Soergel, lec 3: Overview III

1) Toward Koszul duality

2) Geometric interpretation

corrections to 3) in Thm 2 in Sec 1 on 9/17

1) Toward Koszul duality

In Sec 1.2 of Lec 2 we produced an equivalence $\mathcal{O}^{pr}\text{-proj} \xrightarrow{\sim} SMod_{ugr}(W)$. This yields an algebra isom'm

(1) $A := End(\bigoplus_{w\in W} P_0(w\cdot 0)) \xrightarrow{\sim} End_{\mathbb{K}[\mathfrak{h}^*]}(\bigoplus_{w\in W} B_w \otimes_{\mathbb{K}[\mathfrak{h}^*]} \mathbb{K}_0)$

$M := \bigoplus_{w\in W} B_w \otimes_{\mathbb{K}[\mathfrak{h}^*]} \mathbb{K}_0$ is a fin. dim. <u>graded</u> $\mathbb{K}[\mathfrak{h}^*]$-module

So the r.h.s. of (1) is canonically graded: if $M = \bigoplus_{j\in\mathbb{Z}} M_j$ then $End_{\mathbb{K}[\mathfrak{h}^*]}(M)_k = \{\varphi: M\to M \mid \varphi(M_i) \subset M_{i+k}\}$. This equips A with an algebra grading.

It turns out there's a different interpretation of A. Namely consider the principal block $\check{\mathcal{O}}^{pr}$ for Langlands dual Lie algebra $\check{\mathfrak{g}}$. Consider the simple objects $\check{L}_0(w\cdot 0) \in \check{\mathcal{O}}^{pr}$ and the (cohomologically) graded algebra $Ext^*_{\check{\mathcal{O}}^{pr}}(\bigoplus_{w\in W} \check{L}_0(w\cdot 0))$, where $Ext^* = \bigoplus_{i\geqslant 0} Ext^i$

Theorem 1 (Beilinson-Ginzburg, Soergel)

$$A \xrightarrow{\sim} \mathrm{Ext}^*_{\check{\mathcal{O}}^{pr}}\Big(\bigoplus_{w\in W} \check{L}_0(w\cdot 0)\Big) \text{ as graded algebras.}$$

The proof is based on an equivalence of a suitable additive category with $SMod(W)$. The additive category of interest to be denoted by $D^b(\check{\mathcal{O}}^{pr})_{ss}$. This is the full subcategory in $D^b(\check{\mathcal{O}}^{pr})_{ss}$ consisting of the direct sums of $\check{L}_0(w\cdot 0)[j]$, $w\in W$, $j\in\mathbb{Z}$ ($[j]$ is homological shift). The functor

$$\check{\mathbb{V}}_K : M \mapsto \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{D^b(\check{\mathcal{O}}^{pr})}(\check{L}_0(0), M[j]) \quad (\text{"K" for Koszul})$$

take values in $\mathrm{Ext}^*_{\check{\mathcal{O}}^{pr}}(\check{L}_0(0), \check{L}_0(0))$-grmod.

Theorem 2 (Soergel)

1) $\mathrm{Ext}^*_{\check{\mathcal{O}}^{pr}}(\check{L}(0), \check{L}(0)) = \mathbb{C}[\check{\mathfrak{h}}]^{coW}$
2) $\check{\mathbb{V}}_K$ is fully faithful on $D^b(\check{\mathcal{O}}^{pr})_{ss}$
3) & $\check{\mathbb{V}}_K(\check{L}_0(w^{-1}\cdot 0)) = B_{ww_0} \otimes_{\mathbb{K}[\mathfrak{h}^*]} \mathbb{K}_0\langle -\ell(w_0)\rangle$, where w_0 is the longest element in W.

2] **Exercise**: Deduce Thm 1 from Thm 2

Unlike the proofs of Soergel's theorems from Lec 1, which are algebraic and will be explained in this course, the proof of Theorem 2 is geometric. We won't cover it but we'll explain the relevant geometry.

Remark: Theorem 1 is a starting point for studying Koszul duality in Representation theory which is a very important topic.

2) Geometric interpretation

2.1) Strongly equivariant modules

We'll need a general construction. Let $\mathcal{A}$ be a unital associative algebra/$\mathbb{K}$ & F be an algebraic group. Suppose we are given an action of F on $\mathcal{A}$ by automorphisms that is rational meaning that $\forall a \in \mathcal{A}$ $\exists$ subspace $V \subset \mathcal{A}$ s.t. $a \in V$, $\dim V < \infty$, V is F-stable & is an algebraic representation of F. This gives a Lie algebra homomorphism $\mathfrak{f} \to \operatorname{Der}(\mathcal{A})$, $x \mapsto x_{\mathcal{A}}$ by differentiating the F-action.

Definition: A quantum comoment map for $F \curvearrowright \mathcal{A}$ is an F-equivariant linear map $\Phi: \mathfrak{f} \to \mathcal{A}$ s.t.

$$[\Phi(x), a] = x_{\mathcal{A}} a \quad \forall x \in \mathfrak{f}, a \in \mathcal{A}$$

It's automatically a Lie algebra homomorphism.

Example 1 a) Let G be an algebraic group, $F \subset G$ an algebraic subgroup & $\mathcal{A} = \mathcal{U}(\mathfrak{g})$, so that F naturally acts on $\mathcal{A}$. Then the composition $\Phi: \mathfrak{f} \hookrightarrow \mathfrak{g} \hookrightarrow \mathcal{U}(\mathfrak{g})$ is a quantum comoment map.

b) Let Y be a smooth affine variety. We can consider the algebra of differential operators $D(Y)$ on Y. By definition,

- it contains $\mathbb{K}[Y]$ as a subalgebra
- $\mathrm{Vect}(Y) = \mathrm{Der}(\mathbb{K}[Y])$ as Lie subalgebra & left $\mathbb{K}[Y]$-submodule

• the commutation between $\mathbb{K}[Y]$ & $\mathrm{Vect}(Y)$ is $[\xi, f] = \underbrace{\xi . f}_{\text{action of } \xi \text{ on } f \in \mathbb{K}[Y]}$ ($\xi \in \mathrm{Vect}(Y)$, $f \in \mathbb{K}[Y]$).

For example, for $Y = \mathbb{A}^1$ we get the 1st Weyl algebra $\mathbb{K}\langle x, \partial\rangle / (\partial x - x\partial - 1)$

Now suppose F acts on Y. Applying the general construction in the beginning of the section to $F \curvearrowright \mathbb{K}[Y]$ we get an F-equivariant map $x \mapsto x_Y := x_{\mathbb{K}[Y]} : \mathfrak{f} \to \mathrm{Vect}(Y)$. For $x \in \mathfrak{f}$, set $\Phi(x) = x_Y \in \mathrm{Vect}(Y) \subset \mathcal{D}(Y)$.

Note that F acts on $\mathcal{D}(Y)$.

Exercise: Φ is a quantum comoment map.

Now we return to the general situation.

Definition: A weakly F-equivariant $\mathcal{A}$-module is an $\mathcal{A}$-module M together w. a rational F-action s.t. the action map $\mathcal{A} \otimes M \to M$ is F-equivariant. We say M is strongly F-equivariant if $\Phi(x) m = x_M m$

$\forall\, x\in\mathfrak{f}$, $m\in M$ (where $x_M\in End(M)$ comes from differentiating the F-action). Let $\mathcal{A}$-mod^F denote the category of strongly F-equivariant $\mathcal{A}$-modules.

Example 2 a) Let G be a simply connected semisimple algebraic group, $B\subset G$ be a Borel subgroup. We claim that the full subcategory of fin. gen. $\mathcal{U}(\mathfrak{g})$-modules in $\mathcal{U}(\mathfrak{g})$-mod^B is $\mathcal{O}_0$.

First, suppose M is a strongly B-equivariant $\mathcal{U}(\mathfrak{g})$-module. The action of max. torus $H\subset B$ gives a decomposition $M=\bigoplus_{\lambda\in\mathfrak{X}(H)} M_\lambda$, where the summation is over the character lattice $\mathfrak{X}(H)=Hom(H,\mathbb{G}_m)\xrightarrow{\sim}$ [G is simply connected] Λ, & $M_\lambda=\{m\in M \mid t.m=\lambda(t)m\ \forall\, t\in H \Leftrightarrow$ strong equiv. for $H\curvearrowright M$
$xm=\langle\lambda,x\rangle m\ \forall\, x\in\mathfrak{h}\}$, i.e. the same as in def'n for $\mathcal{O}_0$ (Sec 1.1 of Lec 1). The action of $\mathfrak{n}$ comes by differentiating that of N, hence locally nilpotent.

Conversely, the action of $\mathfrak{b}$ on any object $M'\in\mathcal{O}_0$ integrates to B establishing M' as object of $\mathcal{U}(\mathfrak{g})$-mod^B.

2) In the setting of Example 16, $\mathbb{K}[Y]$ is a strongly F-equivariant $\mathcal{D}(Y)$-module (exercise).

Remarks: 1) To recover $\mathcal{O}_\chi$ one replaces Φ w. $\Phi-\chi$, where we view χ as element of $(\mathfrak{b}^*)^B$.

2) One can talk about the sheaf of differential operators $\mathcal{D}_Y$ on any smooth variety Y & the category $\mathcal{D}_Y$-mod of sheaves of $\mathcal{D}_Y$-modules that are quasi-coherent sheaves of $\mathcal{O}_Y$-modules.

3) Suppose $\tilde{Y}\to Y$ be a principal F-bundle (say, locally trivial in the Zariski topology). Then the categories $\mathcal{D}_{\tilde{Y}}\text{-mod}^F$ & $\mathcal{D}_Y$-mod are naturally equivalent (HW 1). Our main example is $\tilde{Y}=G$, a semisimple group, $F=B$ & $Y=G/B$ is the flag variety (non-affine).

2.2) From $\mathcal{D}$-modules to category $\mathcal{O}$

Consider the action of $N\times B$ on G: $(n,b).g=ngb^{-1}$.
We have the following result.

Thm 1: The abelian categories $\{$fin. gen'd $/\mathcal{D}(G)$ objects in $\mathcal{D}(G)\text{-mod}^{N\times B}\}$ & $\mathcal{O}^{pr}$ are equivalent.

Here's one way to see this. The category $\mathcal{D}(G)\text{-mod}^{N\times B}$ is the same as $\mathcal{D}_{G/B}\text{-mod}^N$ of strongly N-equivariant $\mathcal{D}$-modules on G/B (cf. Remark 3). An object, say $\mathcal{F}$, in this category is, in particular, a sheaf on G/B so it makes sense to speak about its global sections $\Gamma(\mathcal{F})\in\Gamma(\mathcal{D}_{G/B})\text{-mod}^N$ (under the equivalence w. $\mathcal{D}(G)\text{-mod}^{N\times B}$, Γ becomes taking the B-invariants).

One can describe $\Gamma(\mathcal{D}_{G/B})$ Lie theoretically. We still have the Lie algebra homomorphism $\mathfrak{g}\to \operatorname{Vect}(G/B)$ coming from $G \curvearrowright G/B$. It gives rise to an associative algebra homomorphism

(1) $\mathcal{U}(\mathfrak{g})\longrightarrow\Gamma(\mathcal{D}_{G/B})$

Recall that Z stands for the center of $\mathcal{U}(\mathfrak{g})$. Let $m_0=\{z\in Z\mid z$ acts by 0 on $\mathbb{K}_{triv}\}$. Set

$$\mathcal{U}_0(\mathfrak{g}) := \mathcal{U}(\mathfrak{g}) / \mathcal{U}(\mathfrak{g})\mathfrak{m}_0$$

For example, for $\mathfrak{g} = \mathfrak{sl}_2$: $\mathcal{U}_0(\mathfrak{sl}_2) = \mathcal{U}(\mathfrak{sl}_2)/(ef + fe + \frac{h^2}{2})$

Thm 2 (Beilinson-Bernstein '81)

i) (1) factors through $\mathcal{U}_0(\mathfrak{g}) \xrightarrow{\sim} \Gamma(\mathcal{D}_{G/B})$.

ii) Thx to i) we have a functor

$$\Gamma: \mathcal{D}_{G/B}\text{-mod} \longrightarrow \mathcal{U}_0(\mathfrak{g})\text{-mod}$$

It's an equivalence of abelian categories.

Corollary: $\Gamma: \mathcal{D}_{G/B}\text{-mod}^N \longrightarrow \mathcal{U}_0(\mathfrak{g})\text{-mod}^N$

Let $\mathcal{O}'^{pr}$ denote the category of Noetherian objects ($\Leftrightarrow$ fin. generated over $\mathcal{U}_0(\mathfrak{g})$) in r.h.s. It does not coincide w. $\mathcal{O}^{pr}$ (the modules in both are fin. generated and have locally nilpotent action of N, but $\mathcal{O}^{pr}$ consists of modules where $\mathfrak{h}$ acts diagonalizably & $\mathfrak{m}_0$ acts locally nilpotently; while $\mathcal{O}'^{pr}$ consists of modules where $\mathfrak{m}_0$ acts by 0, while $\mathfrak{h}$ can be shown to act locally finitely)

Nevertheless, we have the following result (due to Soergel in his diploma work):

Thm 3: We have an equivalence of abelian categories
$\mathcal{O}^{pr} \xrightarrow{\sim} \mathcal{O}'^{pr}$

Combining Thm 3 & Corollary we basically get Thm 1.