$\mathcal{O}$ & Soergel, lec 3: Overview III

1) Toward Koszul duality

2) Geometric interpretation

1) Toward Koszul duality

Last time: $\mathbb{V}: \mathcal{O}^{pr}\text{-proj} \xrightarrow{\sim} SMod_{ugr}(W)$

alg. isom.

(1) $A := End_{\mathcal{O}}\big(\bigoplus_{w\in W} P_o(w\cdot 0)\big) \xrightarrow{\sim} End_{\mathbb{K}[\mathfrak{h}^*]}\big(\underbrace{\bigoplus_{w\in W} B_w \otimes_{\mathbb{K}[\mathfrak{h}^*]} \mathbb{K}_0}\big)$

graded as module $\longrightarrow M$

$M = \bigoplus_{j\in\mathbb{Z}} M_j \rightsquigarrow$ grading on A: deg k part $=$

$= \{\varphi: M \to M \mid \varphi(M_i) \subset M_{i+k}\}$

Different realization of graded alg. A.

Langlands dual $\check{\mathfrak{g}} \rightsquigarrow \check{\mathcal{O}}^{pr} \ni \check{L}_o(w\cdot 0)$ - simples, $w\in W$

$\rightsquigarrow$ graded algebre

$$\underbrace{Ext^*_{\check{\mathcal{O}}^{pr}}}\big(\bigoplus_{w\in W} \check{L}_o(w\cdot 0)\big)$$

$\bigoplus_{i\geq 0} Ext^i \longleftarrow$ algebra grading

Thm 1 (Beilinson-Ginzburg; Soergel)

$$A \xrightarrow{\sim} \mathrm{Ext}^*_{\check{\mathcal{O}}^{pr}}\Big(\bigoplus_{w\in W} \check{L}_0(w\cdot 0)\Big) \text{ (as graded algebras)}$$

Idea: equivalence of additive cats

$$SMod(W) \xleftarrow{\sim} D^b(\check{\mathcal{O}}^{pr})_{ss}$$

$$\Vert$$

$$\{\mathcal{F}\in D^b(\check{\mathcal{O}}^{pr}) \mid \mathcal{F} \simeq \bigoplus_{W\ni w,\, j\in\mathbb{Z}} \check{L}_0(w\cdot 0)[j]\}$$

via $\underbrace{\check{\mathbb{V}}_K}$: $M \longmapsto \bigoplus_{j\in\mathbb{Z}} \mathrm{Hom}_{D^b(\check{\mathcal{O}}^{pr})}(\check{L}_0(0), M[j])$

Koszul

$$\rotatebox{90}{$\in$}$$

$$\mathrm{Ext}^*_{\check{\mathcal{O}}^{pr}}(\check{L}_0(0), \check{L}_0(0))\text{-grmod}$$

Thm 2 (Soergel)

1) $\mathrm{Ext}^*_{\check{\mathcal{O}}^{pr}}(\check{L}_0(0), \check{L}_0(0)) = \mathbb{C}[\check{\mathfrak{h}}]^{coW}$

2) $\check{\mathbb{V}}_K$ is fully faithful on $D^b(\check{\mathcal{O}}^{pr})_{ss}$

3) $\check{\mathbb{V}}(\check{L}_0(w\cdot 0)) = B_w \otimes_{\mathbb{K}[\mathfrak{h}^*]} \mathbb{K}_0$

Exercise: Deduce Thm 1 from Thm 2.

2) Geometric interpretation.

2.1) Strongly equivariant modules

$\mathcal{A}$ assoc. unital alg $/ \mathbb{K}$ (char 0)

F is alg. group.

Let $F \curvearrowright \mathcal{A}$ by autom's & "rational"

: $\forall\, a \in \mathcal{A}$ $\exists$ fin. dim. F-stable $V \subset \mathcal{A}$ s.t.

$a \in V$ & $F \curvearrowright V$ is algebraic repr'n ($\rightsquigarrow \mathfrak{f} \curvearrowright V$)

$\rightsquigarrow$ Lie alg. homom. $\mathfrak{f} \longrightarrow \mathrm{Der}(A): x \mapsto x_{\mathcal{A}}$.

Def'n: A quantum comoment map for $F \curvearrowright \mathcal{A}$ is

F-equiv't linear $\Phi: \mathfrak{f} \longrightarrow \mathcal{A}$ |

$$[\Phi(x), a] = x_{\mathcal{A}} a \quad \forall\, x \in \mathfrak{f},\ a \in \mathcal{A}$$

(Φ is Lie alg. homom.)

Example 1)

a) $F \subset G$ alg. groups, $\mathcal{A} = \mathcal{U}(\mathfrak{g}) \rightsquigarrow F \curvearrowright \mathcal{U}(\mathfrak{g})$

Φ = composition $\mathfrak{f} \hookrightarrow \mathfrak{g} \hookrightarrow \mathcal{U}(\mathfrak{g})$, quant. comoment map.

6) Y smooth affine variety $\rightsquigarrow$ algebra $D(Y)$ of differential operators:

- $\mathbb{K}[Y] \subset D(Y)$ subalgebra
- $\mathrm{Vect}(Y) = \mathrm{Der}_{\mathbb{K}}(\mathbb{K}[Y]) \subset D(Y)$ as Lie subalgebra & as left $\mathbb{K}[Y]$-submodule
- $\xi \in \mathrm{Vect}(Y)$, $f \in \mathbb{K}[Y]$: $\underbrace{[\xi, f]}^{\text{in } D(Y)} = \underbrace{\xi . f}_{\text{action}}$ $(\in \mathbb{K}[Y])$

e.g. $Y = \mathbb{A}^1 \Rightarrow D(Y) = \mathbb{K}\langle x, \partial\rangle / (\partial x - x\partial - 1)$.

Let $F \curvearrowright Y \rightsquigarrow F \curvearrowright \mathbb{K}[Y] \rightsquigarrow \mathfrak{f} \longrightarrow \mathrm{Vect}(Y)$, $x \mapsto x_{\mathbb{K}[Y]} =: x_Y$

$\Phi : x \mapsto x_Y \in \mathrm{Vect}(Y) \subset D(Y)$.

Exercise: Φ is quantum comoment map.

Def'n: Weakly F-equivariant $\mathcal{A}$ module M is $\mathcal{A}$-module + rational F-rep'n M s.t. $\mathcal{A} \otimes M \to M$ is F-linear. If we have Φ can talk about strongly F-equiv't modules M = weakly equiv't +

$\Phi(x)m = \underbrace{x_M}_{}\, m \quad \forall\, x \in \mathfrak{f},\ m \in M$

image of x under $\mathfrak{f} \to End(M) \leftsquigarrow F \curvearrowright M$.

Category of strongly equiv. modules $\mathcal{A}\text{-mod}^F$

Example 2: a) G simply connected semisimple gr'p, $B \subset G$ Borel $\rightsquigarrow \mathcal{U}(\mathfrak{g})\text{-mod}^B$

$$\cup$$

$$\{\text{fin. gen'd } \mathcal{U}(\mathfrak{g})\text{-modules}\} \xrightarrow{\sim} \mathcal{O}_0$$

Why: $M \in \mathcal{U}(\mathfrak{g})\text{-mod}^B$

max. torus $H \subset B \rightsquigarrow H \curvearrowright M$

$$\Big\Updownarrow$$

$$M = \bigoplus_{\lambda \in \mathfrak{X}(H)} M_\lambda$$

$\mathfrak{X}(H) = Hom(H, \mathbb{G}_m)$ - character lattice

$\downarrow s$

Λ ; $M_\lambda = \{m \in M \mid t.m = \lambda(t)m \ \forall\, t \in H\}$

$\Updownarrow$ strong equiv. for $H \curvearrowright M$

$x m = \langle \lambda, x \rangle m \ \forall\, x \in \mathfrak{h}\}$

$\mathfrak{n}$ acts locally nilp. b/c by strong equiv for $N \curvearrowright M$ this action is diff'l of $N \curvearrowright M$ (unip. group).

Conversely, $M' \in \mathcal{O}_0 \Rightarrow$ can integrate $\mathfrak{b} \curvearrowright M$ to $B \curvearrowright M$ turning M' into object of $\mathcal{U}(\mathfrak{g})\text{-mod}^B$.

6) $F \curvearrowright D(Y)$; $\mathbb{K}[Y]$ is strongly F-equivariant $D(Y)$-module (exercise)

Remarks: 1) To recover $\mathcal{O}_\chi$ ($\chi \in \mathfrak{h}^*$) replace Φ w. $\Phi - \chi$ (also comoment map)

2) We considered smooth <u>affine</u> Y. For general smooth Y, have the <u>sheaf</u> D_Y of diff'l operators & category $\mathcal{D}_Y\text{-mod}$ of sheaves of $\mathcal{D}_Y$-modules that are quasi-coherent over $\mathcal{O}_Y \subset \mathcal{D}_Y$.

3) $\tilde{Y} \longrightarrow Y$, a principal F-bundle (e.g. $\tilde{Y} = G$, $F = B$, $Y = G/B$ flag variety). Then $\mathcal{D}_{\tilde{Y}}\text{-mod}^F \xrightarrow{\sim} D_Y\text{-mod}$ (HW1).

2.2) $\mathcal{O}$ vs equivariant $\mathcal{D}$-modules

$N\times B \curvearrowright G$ $(n,b)g = ngb^{-1} \rightsquigarrow N\times B \curvearrowright \mathcal{D}(G)$ w. Φ.

Thm 1: $\{\text{fin. gen'd } \mathcal{D}(G)\text{-mods in } \mathcal{D}(G)\text{-mod}^{N\times B}\} \xrightarrow{\sim} \mathcal{O}^{pr}$ as abelian categories

basically remark 3

How to prove this: $\mathcal{D}(G)\text{-mod}^{N\times B} \xrightarrow{\sim} \mathcal{D}_{G/B}\text{-mod}^N$

$\mathcal{F} \in \mathcal{D}_{G/B}\text{-mod}^N$ is sheaf on $G/B \rightsquigarrow$ global sections $\Gamma(\mathcal{F})$ is N-equiv't module over $\Gamma(\mathcal{D}_{G/B})$

how to describe?

$G \curvearrowright G/B \rightsquigarrow \mathfrak{g} \longrightarrow \mathrm{Vect}(G/B)$ (Lie alg. homom.) $\rightsquigarrow$

(1) $\mathcal{U}(\mathfrak{g}) \longrightarrow \Gamma(\mathcal{D}_{G/B})$ (of assoc. algebras)

$\mathcal{Z}$ = center of $\mathcal{U}(\mathfrak{g})$

$\cup$

$m_0 = \{z \in \mathcal{Z} \mid z \text{ acts by } 0 \text{ on } \mathbb{K}_{triv}\}$

$\mathcal{U}_0(\mathfrak{g}) := \mathcal{U}(\mathfrak{g})/\mathcal{U}(\mathfrak{g})m_0$

e.g. $\mathfrak{g} = \mathfrak{sl}_2$: $\mathcal{U}_0(\mathfrak{sl}_2) = \mathcal{U}(\mathfrak{sl}_2)/(ef+fe+\frac{h^2}{2})$.

Thm 2 (Beilinson-Bernstein '81)

i) (1) factors through $\mathcal{U}_0(\mathfrak{g}) \xrightarrow{\sim} \Gamma(\mathcal{D}_{G/B})$

ii) i) $\rightsquigarrow \Gamma : \mathcal{D}_{G/B}\text{-mod} \longrightarrow \mathcal{U}_0(\mathfrak{g})\text{-mod}$

It's equivalence of abelian cat's.

Corollary: $\Gamma : \mathcal{D}_{G/B}\text{-mod}^N \longrightarrow \mathcal{U}_0(\mathfrak{g})\text{-mod}^N$

$\mathcal{O}'^{pr}$ = full subcat of fin. gen.'d ($\Leftrightarrow$ Noetherian) in $\mathcal{U}_0(\mathfrak{g})\text{-mod}^N$

not the same as $\mathcal{O}^{pr}$ (do not consist of same modules!)

Thm 3 (Soergel): Have equivalence of abelian cat's

$$\begin{tikzcd}
\mathcal{O}^{pr} \arrow[rr, "\sim"] & & \mathcal{O}'^{pr} \\
L_0(w\cdot 0) \arrow[u, phantom, "\ni"] \arrow[rr, mapsto] & & L_0(w^{-1}\cdot 0) \arrow[u, phantom, "\in"] \\
& \{\text{fin. gen'd } \mathcal{U}(\mathfrak{g})\text{-}\mathcal{U}_0(\mathfrak{g})\text{-bimodules s.t.} \arrow[uul, bend left] \arrow[uur, bend right] &
\end{tikzcd}$$

Harish-Chandra bimodules^2 $\longrightarrow$ {fin. gen'd $\mathcal{U}(\mathfrak{g})$-$\mathcal{U}_0(\mathfrak{g})$-bimodules s.t.

- $\mathfrak{m}_0$ acts locally nilpotently
- adjoint $\mathfrak{g}$-action is locally finite}