O & Soergel, lec 4: Overview IV

1) Crash course on equivariant $\mathcal{D}$-modules

Corrections to homological shifts from 9/17

1) Crash course on equivariant $\mathcal{D}$-modules

1.1) Basics

Let Y be a smooth variety & F is an algebraic group acting on Y. Two situations we'll care about:

1) $Y=G/B$, $F=N$ (or $Y=G$, $F=N\times B$)

2) $Y=G/B$, $F=B$ (or $Y=G$, $F=B\times B$).

To $F \curvearrowright Y$ we assign the (bounded) derived category $D^b(Y/F)$ of F-equivariant $\mathcal{D}$-modules on Y. It has a t-structure whose heart is the abelian category of F-equivariant $\mathcal{D}$-modules on Y, $\mathcal{D}_Y\text{-mod}^F$ (introduced in Sec 2.1 of Lec 3 in the case when Y was affine). In some cases ($F=\{1\}$ or Case 1) $D^b(Y/F)\simeq D^b(\mathcal{D}_Y\text{-mod}^F)$ but in general (e.g. $F=\mathbb{G}_m$, $Y=pt$) this is not the case.

1.2) Classification of irreducible objects

Our goal is to classify irreducible objects in the abelian category $\mathcal{D}_Y\text{-mod}^F$ under the following

(*) F has fin. many orbits on Y: $Y = \coprod_{i=1}^{k} Y_i$, $Y_i = F/F_i$

Let $F_i^\circ \subset F_i$ be the connected component of $1 \in F_i \rightsquigarrow$ finite group F_i/F_i°.

Fact: The irreducible objects in $\mathcal{D}_Y\text{-mod}^F$ are in bijection with $\coprod_{i=1}^{k} \mathrm{Irrep}(F_i/F_i^\circ)$, $(i, \sigma_i) \mapsto IC(Y_i, \sigma_i)$

Examples: 1) If $\overline{Y_i} = Y$, $\sigma_i = \mathrm{triv}$, then $IC(Y_i, \mathrm{triv}) = \mathcal{O}_Y$

2) Let $Y = G/B$ & $F = N$ or B. We have Bruhat decomposition: $G = \coprod_{w \in W} N\dot{w}B$, where $\dot{w} \in W = N_G(H)/H$ is a lift of w to $N_G(H)$, so both cases satisfy (*). Let $N_w := \{n \in N \mid n\dot{w}B = \dot{w}B\}$, it's unipotent $\Rightarrow N_w^\circ = N_w \Rightarrow \mathrm{Irr}(\mathcal{D}_{G/B}\text{-mod}^N) \xrightarrow{\sim} W$ (as we should have thx to the equivalence w. $\mathcal{O}^{pr}$ from Sec 2.2 in Lec 3). And if $F = B$, then the stabilizers are $H \ltimes N_w$, also connected.

1.3) Construction of irreducible objects.

We need tools for understanding the objects $IC(Y_i, triv)$. Inside $\mathcal{D}_Y\text{-mod}^F$ we can consider the full subcategory of "holonomic modules". E.g. $\mathcal{O}_Y$ is holonomic. If Y satisfies (*), then $\mathcal{F} \in \mathcal{D}_Y\text{-mod}^F$ is holonomic $\Leftrightarrow$ $\mathcal{F}$ has finite length. In general, let $D^b_{hol}(Y/F)$ be the full subcategory of complexes with holonomic cohomology.

An advantage of D^b_{hol} is that it has 6 functor formalism. E.g. $\forall$ F-equivariant $f: Y_1 \to Y_2$ we have:

1) The pullback functor $f^*: D^b_{hol}(Y_2) \longrightarrow D^b_{hol}(Y_1)$ (w. usual functoriality: $(fg)^* = g^* f^*$). We have $f^* \mathcal{O}_{Y_2} = \mathcal{O}_{Y_1}[\dim Y_2 - \dim Y_1]$

2) Its right adjoint, pushforward functor f_*

Assume (*) in Sec 1.2. Consider the inclusion $\iota_i: Y_i \hookrightarrow Y$ (locally closed). Then:

a) $\iota_{i,*}\mathcal{O}_{Y_i} \in D^b_{hol}(Y/F)$ is in cohom. deg $\geqslant 0$

b)
$$IC(Y_i, triv) \hookrightarrow \mathcal{H}^0(\iota_{i,*}\mathcal{O}_{Y_i}) \qquad (1)$$
it's supported (as a quasi-coherent sheaf) on $\overline{Y}_i$; coker of (1) & $\mathcal{H}^j(\iota_{i,*}\mathcal{O}_{Y_i})$ are supported on $\overline{Y}_i \setminus Y_i$ $\forall j > 0$.

c) assume $\overline{Y}_i$ is smooth & $\iota_i : \overline{Y}_i \hookrightarrow Y$ denote the inclusion. Then $IC(Y_i, triv) = \iota_{i,*} \mathcal{O}_{\overline{Y}_i}$.

We will need the following result, equivariant case of the Decomposition theorem of Beilinson-Bernstein-Deligne-Gabber.

Thm: Let $\varphi : \tilde{Y}_i \to \overline{Y}_i$ be F-equiv't projective morphism & $\tilde{Y}_i$ is smooth. Then $\varphi_* \mathcal{O}_{\tilde{Y}_i} \simeq \bigoplus_{Y_j \subset \overline{Y}_i, \sigma_j} IC(Y_j, \sigma_j)[?]^{\oplus ?}$ Moreover if φ is an isomorphism over Y_i, then the only term w. $Y_j = Y_i$ is $IC(Y_i, triv)$ in cohomological deg 0 & with multiplicity 1.

Rem*: "Riemann-Hilbert correspondence" gives a functor from $D^b_{hol}(Y/F)$ to the category of equivariant constructible sheaves $D^b_c(Y/F)$. It sends $\mathcal{O}_Y$ to the rk 1 constant sheaf $\underline{\mathbb{C}}_Y$ & intertwines the functors f^*, f_* w. usual pullback & pushforwards for sheaves of vector spaces.

1.3) Equivariant cohomology.

We'll be interested in Ext's from $\mathcal{O}_Y$ in $D^b(Y/F)$. For this we need an account on the equivariant cohomology. Here $\mathbb{K}=\mathbb{C}$, so any variety is a topological space. Let X be a topological space w. (topological) F-action. To this action we assign the **equivariant cohomology** $H^*_F(X):=H^*((E\times X)/F)$, where:

- E is contractible topological space w. topol. F-action
- & $\exists$ topological principal F-bundle $E\longrightarrow E/F$.

If $\exists$ topological principal F-bundle $X\to X/F$, then $(E\times X)/F\longrightarrow X/F$ is a loc. trivial bundle w. contractible fibers, so $H^*_F(X)=H^*(X/F)$. So, in general, one views $H^*_F(X)$ is the "correct cohomology" of X/F. It's a graded $H^*_F(pt)$-algebra (via pullback w.r.t. $X\to pt$)

Special cases:

1) F is unipotent so isomorphic to an affine space $\Rightarrow$ can take $E=F$ so $H^*_F(X)=H^*(X)$.

2) $F = \mathbb{G}_m$. Then can take $E = \mathbb{C}^\infty \setminus \{0\}$, where $\mathbb{C}^\infty = \{(z_i)_{i\geq 0} \mid i \in \mathbb{Z}_{\geq 0}$ & all but fin. many $z_i = 0\}$ so

$E/F = \mathbb{CP}^\infty$ & $H_F^*(pt) = H^*(\mathbb{CP}^\infty) = \mathbb{C}[x], x \in H^2(\mathbb{CP}^\infty)$

More generally, $H_H^*(pt)$ is canonically $\mathbb{C}[\mathfrak{h}]$.

3) $H_B^*(X) \xrightarrow{\sim} H_H^*(X)$ (exercise).

Remarks: 1) Generalizing 2): if G is a connected reductive group, then $H_G^*(pt) = \mathbb{C}[\mathfrak{g}]^G$ w. $\deg \mathfrak{g}^* = 2$. From here one can deduce that for any F, $H_F^*(pt)$ is in even degrees (and hence genuinely commutative)

2) Note that $(E\times X)/F \to E/F$ is a topologically trivial fiber bundle w. fiber X. This gives a pullback homomorphism $H_F^*(X) \to H^*(X)$ factoring through

$$H_F^*(X) \otimes_{H_F^*(pt)} \mathbb{C} \longrightarrow H^*(X) \qquad (2)$$

If $H^*(X)$ is in even degrees then the spectral sequence for the cohomology of total space of a fiber bundle degenerates at E_2 and (2) is iso. This will be the case for X in the next section.

1.4) Case of flag & Bott-Samelson varieties.

Now we'll give examples of computation of $H^*_F(Y)$.

I) $H^*(G/B) = H^*_B(G)$ is an $H^*_B(pt) = \mathbb{C}[\mathfrak{h}]$-algebra. One can show it's $\mathbb{C}[\mathfrak{h}]^{coW}$ ($= \mathbb{C}[\mathfrak{h}]/\mathbb{C}[\mathfrak{h}]\mathbb{C}[\mathfrak{h}]^W_+$, where $\mathbb{C}[\mathfrak{h}]^W_+ = \{f \in \mathbb{C}[\mathfrak{h}]^W \mid f(0)=0\}$); $\mathbb{C}[\mathfrak{h}]$, $\mathbb{C}[\mathfrak{h}]^{coW}$ are graded so that $\deg \mathfrak{h}^* = 2$ (cf. 2) above).

II) Similarly, $H^*_B(G/B) = H^*_{B\times B}(G)$ is a $\mathbb{C}[\mathfrak{h}]\otimes\mathbb{C}[\mathfrak{h}]$-algebra. It is $\mathbb{C}[\mathfrak{h}]\otimes_{\mathbb{C}[\mathfrak{h}]^W}\mathbb{C}[\mathfrak{h}]$.

III) Recall that S denotes the set of simple reflections in W. To $\underline{w} := (s_1,\ldots,s_k)\in S^k$ we assign the Bott-Samelson variety $\mathcal{BS}_{\underline{w}}$ as follows. To $s\in S$ we assign the minimal parabolic $P_s \subset G$ containing B. Let $\mathcal{BS}_{\underline{w}}$ be the quotient of $P_{s_1}\times\ldots\times P_{s_k}$ by the following B^k-action: $(b_1,\ldots b_k).(p_1,\ldots p_k) = (p_1 b_1^{-1}, b_1 p_2 b_2^{-1}, \ldots, b_{k-1} p_k b_k^{-1})$. Denote the orbit of $(p_1,\ldots p_k)$ by $[p_1,\ldots p_k]$. We have the following maps:

$$\mathcal{BS}_{(s_1,\ldots s_k)} \longrightarrow \mathcal{BS}_{(s_1,\ldots s_{k-1})},\ [p_1,\ldots p_k] \to [p_1,\ldots p_{k-1}] \quad (3)$$

$$\mathcal{BS}_{(s_1,\ldots s_k)} \longrightarrow G/B,\quad [p_1,\ldots p_k] \mapsto p_1\ldots p_k B. \quad (4)$$

(4) establishes $H^*_B(\mathcal{BS}_{\underline{w}})$ as a $H^*_B(G/B)$-algebra.

(3) is a $\mathbb{P}^1$-bundle. This allows to compute $H^*_B(\mathcal{BS}_{\underline{w}})$ inductively: it is $BS_{\underline{w}}\langle -K\rangle$, the Bott-Samelson bimodule over $\mathbb{C}[\mathfrak{h}]\otimes_{\mathbb{C}[\mathfrak{h}]^W}\mathbb{C}[\mathfrak{h}]$. And $H^*(\mathcal{BS}_{\underline{w}}) = BS_{\underline{w}}\otimes_{\mathbb{C}[\mathfrak{h}]}\mathbb{C}_0\langle -K\rangle$, a graded $H^*(G/B)=\mathbb{C}[\mathfrak{h}]^{coW}$-module.

Remark: Here's a motivation to consider $\mathcal{BS}_{\underline{w}}$'s: pick a reduced expression $\underline{w}=(s_1,\dots s_\ell)$ for w. The morphism $\varphi_{\underline{w}}$: $\mathcal{BS}_{\underline{w}}\to G/B$ from (4) has image $\overline{BwB/B}$ & is a resolution of singularities of its image.

1.5) $Ext^*_{D^b(Y/F)}(\mathcal{O}_Y, ?)$

Fact: For any smooth Y w. F-action, we have a graded algebra isomorphism $Ext^*_{D^b(Y/H)}(\mathcal{O}_Y,\mathcal{O}_Y)=H^*_F(Y)$. In particular, $\forall\ \mathcal{F}\in D_Y\text{-mod}^F$, $Ext^*_{D^b(Y/H)}(\mathcal{O}_Y, F)$ is a graded right $H^*_F(Y)$-module.

Q: Under assumption $(*)$ from Sec 1.2, how to compute $Ext^*_{D^b(Y/F)}(\mathcal{O}_Y, IC(Y_i, triv))$?

Easy case: $\overline{Y}_i$ is smooth. For $\iota_i : \overline{Y}_i \hookrightarrow Y$ we have $\mathrm{Ext}^*(\mathcal{O}_Y, IC(Y_i, \mathrm{triv})) = $ [3) in Sec 1.2] $= \mathrm{Ext}^*(\mathcal{O}_Y, \iota_{i*}\mathcal{O}_{\overline{Y}_i}) =$ [adjunction] $= \mathrm{Ext}^*(\iota_i^*\mathcal{O}_Y, \mathcal{O}_{\overline{Y}_i}) = \mathrm{Ext}^*(\mathcal{O}_{\overline{Y}_i}[\dim Y - \dim Y_i], \mathcal{O}_{\overline{Y}_i}) = H_F^*(\overline{Y}_i)\langle \dim Y_i - \dim Y\rangle$ w. its natural graded $H_F^*(Y)$-module structure

General case: Define the equivariant intersection cohomology $IH_F^*(\overline{Y}_i)$ as $\mathrm{Ext}^*_{D^b(Y/H)}(\mathcal{O}_Y, IC(Y_i, \mathrm{triv})[\dim Y])$ so that if $\overline{Y}_i$ is smooth, we get $H_F^*(\overline{Y}_i)\langle \dim Y_i\rangle$.

Corollary (of BBDG Thm in Sec 1.2) Under assumption (*), if $\widetilde{Y}_i$ is an H-equivariant resolution of singularities of $\overline{Y}_i$, then we have a graded $H_F^*(Y)$-module isomorphism:

$$H_F^*(\widetilde{Y}_i)\langle \dim Y_i\rangle \simeq IH_F^*(\overline{Y}_i) \oplus \bigoplus_{j \mid \overline{Y}_j \subsetneq \overline{Y}_i, \sigma_j} \mathrm{Ext}^*(\mathcal{O}_Y, IC(Y_j, \sigma_j)[?])^{\oplus ?}$$

Example: Let $F = B$, $Y = G/B$, so $H_B^*(G/B) = \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}[\mathfrak{h}]^W} \mathbb{C}[\mathfrak{h}]$

Let $Y_i = BwB/B$ ($\dim Y_i = \ell(w)$), $\widetilde{Y}_i = \mathcal{BS}_{\underline{w}}$, $\varphi = \varphi_{\underline{w}}$. Let $\mathcal{B}'_w :=$ $IH_B^*(\overline{Y}_i)$. Then $\mathcal{BS}_{\underline{w}} \simeq \mathcal{B}'_w \oplus \bigoplus_{u \mid \ell(u) < \ell(w)} \mathcal{B}'_u\langle ?\rangle^{\oplus ?}$. Later we'll see $\mathcal{B}_w' = \mathcal{B}_w$.