# $\mathcal{O}$ & Soergel, lec 5: Basics on category $\mathcal{O}$, I

1) Toward Koszul duality, finished

2) Deformed category $\mathcal{O}$

## 1) Toward Koszul duality, finished

In Sec 1 of Lec 3 we've stated the following result (we now use $\mathfrak{g}$ instead of $\check{\mathfrak{g}}$). Consider the functor

$\mathbb{V}_K: M \mapsto \bigoplus_{j\in\mathbb{Z}} Hom_{D^b(\mathcal{O}^{pr})}(L_0(0), M[j])$:

$\{\bigoplus L_0(?)[?]\} =: D^b(\mathcal{O}^{pr})_{ss} \longrightarrow Ext^*(L_0(0), L_0(0))^{opp}$-grmod.

**Theorem** (Soergel)

1) $Ext^*_{\mathcal{O}^{pr}}(L_0(0), L_0(0)) = \mathbb{C}[\mathfrak{h}]^{coW}$

2) $\mathbb{V}_K$ is fully faithful on $D^b(\mathcal{O}^{pr})_{ss}$

3) & $\mathbb{V}_K(L_0(w^{-1}\cdot 0)) = B_{ww_0} \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C}_0 \langle -\ell(w_0)\rangle$ ($w_0$ is the longest element in $W$).

Sketch of proof:

Thanks to Thm 1 in Sec 2.2 of Lec 3,

$$\mathcal{O}^{pr} \xrightarrow{\sim} \mathcal{D}\text{-mod}^N_{G/B} \qquad (1)$$

One can show that:

i) Under (1), $\mathcal{L}_0(w^{-1}w_0\cdot 0)\mapsto IC(BwB/B, triv)$, in particular $\mathcal{L}_0(0)\mapsto \mathcal{O}_{G/B}$.

ii) $D^b(N\backslash G/B)\xleftarrow{\sim} D^b(\mathcal{D}_{G/B}\text{-mod}^N)$

So we are computing Ext's in $D^b(N|G/B)$. 1) follows from $Ext_{D^b(N\backslash G/B)}(\mathcal{O}_{G/B},\mathcal{O}_{G/B})\xrightarrow{\sim} H^*_N(G/B)=H^*(G/B)=\mathbb{C}[\mathfrak{h}]^{coW}$.
2) is a general result of Ginzburg from 1991. To prove 3) we use a form of the BBDG decomposition Thm (Corollary in Sec 1.5, cf. Example in that section):

$$BS_{\underline{w}}\otimes_{\mathbb{C}[\mathfrak{h}]}\mathbb{C}_0\simeq IH^*_N(\overline{BwB}/B)\oplus\bigoplus_{u|\ell(u)<\ell(w)} IH^*_N(\overline{BuB}/B)\langle ?\rangle^{\oplus ?}$$

By 2) $IH^*_N(\overline{BwB}/B)=\mathbb{V}_K(\mathcal{L}_0(?)[?])$ is indecomposable as a graded $\mathbb{C}[\mathfrak{h}]^{coW}$-module. So it must be $B_w\otimes_{\mathbb{C}[\mathfrak{h}]}\mathbb{C}_0$ by what was explained in Sec 1.2 of Lec 2. This finishes the proof. $\square$

**Remark**: One can show $IH^*_B(\overline{BwB}/B)\otimes_{\mathbb{C}[\mathfrak{h}]}\mathbb{C}_0\simeq IH^*_N(\overline{BwB}/B)$ similarly to what happens w. the usual cohomology (cf. Remark in Sec 1.3 of Lec 4). This shows

$$IH^*_B(\overline{BwB}/B)\simeq B_w.$$

## 2) Deformed category $\mathcal{O}$

### 2.1) Definition

The base field is $\mathbb{C}$.

Let $\mathfrak{g} \supset \mathfrak{b} = \mathfrak{h} \ltimes \mathfrak{n}$ be as before. Let $A$ be a Noetherian $S(\mathfrak{h})$-algebra, we write $\mathcal{X}$ for the corresponding map $\mathfrak{h} \to A$ so that $\mathcal{X} \in \mathfrak{h}^* \otimes_{\mathbb{C}} A$.

The following generalizes the definition from Sec 1.1 of Lec 1.

**Definition 1**: Let $\mathcal{O}_A$ (or $\mathcal{O}_{A,\mathcal{X}}$ if we need to indicate $\mathcal{X}$ explicitly) be the full subcategory in $\mathcal{U}(\mathfrak{g}) \otimes A$-mod consisting of all modules $M$ s.t.:

1) $M$ is fin. gen. / $\mathcal{U}(\mathfrak{g}) \otimes A$

2) $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$, where $M_\lambda = \{m \in M \mid xm = \langle \lambda, x \rangle m + \mathcal{X}(x) m$ $\forall x \in \mathfrak{h}\}$, this automatically a $\mathcal{U}(\mathfrak{h}) \otimes A$ - submodule.

3) $\mathfrak{n}$ acts locally nilpotently on $M$.

**Definition 2**: To $\lambda \in \Lambda$ we assign the Verma module $\Delta_A(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes A / \mathcal{U}(\mathfrak{g}) \otimes A \,(x - \langle \lambda, x \rangle - \mathcal{X}(x), y \mid x \in \mathfrak{h}, y \in \mathfrak{n})$. It has the following universal property:

$$\mathrm{Hom}_{\mathcal{O}_A}(\Delta_A(\lambda), M) \xrightarrow{\sim} M_\lambda^{\mathfrak{n}} := \{m \in M_\lambda \mid ym = 0\ \forall y \in \mathfrak{n}\} \quad (2)$$

The following corollary of (2) will be important

**Lemma:** $\forall M \in \mathcal{O}_A \setminus \{0\}$ $\exists \lambda$ & non-zero $\Delta_A(\lambda) \to M$.

Proof:

Take nonzero $m' \in M_{\lambda'}$ & minimal $r$ s.t. $y_1 \dots y_r m' = 0$ $\forall$ $y_1, \dots y_r \in \mathfrak{n}$. Then we can find root vectors $y_i \in \mathfrak{g}_{\beta_i} \subset \mathfrak{n}$ $(i = 1, \dots r-1)$ s.t. $m = y_1 \dots y_{r-1} m' \neq 0$. Set $\lambda = \lambda' + \sum_{i=1}^{r-1} \beta_i$. Then $0 \neq m \in M_\lambda^{\mathfrak{n}} \rightsquigarrow$ non-zero $\Delta_A(\lambda) \to M$ by (2) $\square$

**Remark.** The choice of $A$ we'll actually need is the completion $\mathbb{C}[[\mathfrak{h}^*]]$ of $\mathbb{C}[\mathfrak{h}^*] = S(\mathfrak{h})$ at $0$ with a natural map $\mathfrak{h} \to A$, as well as its localization. These are very important for proofs of Soergel's thms from Lec 1.

**Important exercise:** $\forall$ $U(\mathfrak{g}) \otimes A$-submodule $N \subset M \Rightarrow N = \bigoplus_{\lambda \in \Lambda} N \cap M_\lambda$.

Also $\forall \lambda \in \Lambda$, the functor $\mathcal{O}_A \to A\text{-mod}$, $M \mapsto M_\lambda$, is exact.

## 2.2) Finiteness properties.

**Lemma:** $\mathcal{U}(\mathfrak{g})\otimes A$ is (left & right) Noetherian.

Proof:

**Exercise:** If $\mathcal{A}$ is a $\mathbb{Z}_{\geq 0}$-filtered associative algebra s.t. the associated graded $\operatorname{gr}\mathcal{A}$ is Noetherian, then $\mathcal{A}$ is Noetherian.

We apply this to $\mathcal{A}=\mathcal{U}(\mathfrak{g})\otimes A$ with PBW filtration (where $A$ is in deg $0$) so that $\operatorname{gr}\mathcal{A}=S(\mathfrak{g})\otimes A$, Noetherian b/c it's a fin. gen. commut've $A$-algebra $\square$

**Exercise:** $\mathcal{O}_A$ is an abelian category.

**Proposition:** Let $M\in\mathcal{O}_A$. Then $\exists$ filtrations by submodules: $\{0\}=M_0\subsetneq M_1\subsetneq\ldots\subsetneq M_k=M$ & $\lambda_1,\ldots\lambda_k\in\Lambda \mid \Delta(\lambda_i)\twoheadrightarrow M_i/M_{i-1}\ \forall i.$

Proof:

Take $M_1$ to be the image of a homomorphism in Lemma from Sec 2.1, then apply Lemma to $M/M_1$ to construct $M_2/M_1$ etc. This terminates b/c $M$ is Noetherian $\square$

**Corollary**: Let $M, M' \in \mathcal{O}_A$. Then:

1) $M_\lambda$ is fin. gen'd over $A$ $\forall \lambda \in \Lambda$
2) $\mathrm{Hom}_{\mathcal{O}_A}(M', M)$ is fin. gen'd over $A$.

Proof:

1) Proposition & Important exercise from Sec 2.1 reduce this to the case $M = \Delta(\lambda)$, which is an exercise.

2) Again we reduce to $M' = \Delta(\lambda)$, where $\mathrm{Hom}_{\mathcal{O}_A}(M', M) = M_\lambda^{\mathfrak{n}} \subset M_\lambda$ by (2). We are done by part 1) $\square$

## 2.3) Infinitesimal block decomposition

For this section we assume $A$ is local. Let $\mathfrak{m} \subset A$ be the maximal ideal & $\mathbb{K} := A/\mathfrak{m}$ Define an equivalence $\sim_\chi$ on $\Lambda$ by: $\lambda_1 \sim_\chi \lambda_2$ if $\exists\, w \in W \mid$

$$\lambda_1 + \chi - w \cdot (\lambda_2 + \chi) \,(\in \mathfrak{h}^* \otimes A) \text{ is in } \mathfrak{h}^* \otimes \mathfrak{m}$$

E.g. if $\operatorname{im} \chi \subset \mathfrak{m}$, then the equivalence classes are the orbits of the $(W, \cdot)$-action. In general, $\forall$ equiv. class is finite.

**Definition**: For an equivalence class $\Xi$ we define the infinitesimal block $\mathcal{O}_A^{\Xi}$ as the full subcategory of all $M \in \mathcal{O}_A$ that admit a filtration as in Prop'n from Sec 2.2 w. $\lambda_i \in \Xi$ $\forall$ $i = 1, .. k$.

Theorem: $\mathcal{O}_A = \bigoplus_{\Xi \in \Lambda/\sim_{\mathcal{X}}} \mathcal{O}_A^{\Xi}$

Note that if $A = \mathbb{K}$, we recover the decomposition in Sec 1.2 of Lec 1.

Proof of Theorem:

Step 1: we show that if $\Xi_1 \neq \Xi_2$ & $M_i \in \mathcal{O}_A^{\Xi_i}$ $(i=1,2)$, then $\mathrm{Hom}_{\mathcal{O}_A}(M_1, M_2) = 0$. Inequality $\Xi_1 \neq \Xi_2$ & HC isom'm ($Z \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^*]^{(W,\cdot)}, z \mapsto HC_z$) $\Rightarrow$

$\exists$ $z \in Z$ $| HC_z(\lambda_1 + \mathcal{X}) - HC_z(\lambda_2 + \mathcal{X}) \notin \mathfrak{m}$

Set $a := HC_z(\lambda_1 + \mathcal{X}) \in A \Rightarrow z - a$ acts by $0$ on $\Delta_A(\lambda_1)$ $\forall$ $\lambda_1 \in \Xi_1$ and invertibly on $\Delta_A(\lambda_2)$ & its quotients $\forall$ $\lambda_2 \in \Xi_2$ (here we use that $A$ is local) $\Rightarrow z - a$ acts nilpotently on $M_1$ & invertibly on $M_2$ (b/c of filtrations by quotients of Vermas). So $\mathrm{Hom}_{\mathcal{O}_A}(M_1, M_2) = 0$.

Step 2: We claim that every SES

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$$

with $M_1, M_2$ as in Step 1 splits. Indeed, in the notation of Step 1 pick $n>0$ s.t. $(z-a)^n$ acts by $0$ on $M_1$. Then $(z-a)^n M \subset M$ maps isomorphically onto $M_2$ splitting SES.

Step 3: Since every $M$ admits a filtration by quotients of $\Delta(\lambda_i)$ (objects of suitable $\mathcal{O}_\Lambda^{\Xi_i}$), the claim of Thm follows from Step 1,2 (exercise). $\square$