O & Soergel, lec 6: Basics on category $\mathcal{O}$, II

1) Finite length

2) Projective objects & projective functors

0) Recap

Let A be a local Noetherian $S(\mathfrak{h})$-algebra (w. the corresponding element $\chi\in\mathfrak{h}^*\otimes A$). Let $\mathfrak{m}\subset A$ be the max. ideal & $\mathbb{K}=A/\mathfrak{m}$. In Sec 2.3 of Lec 5 we introduced an equivalence relation $\sim_\chi$ on Λ:

$$\lambda_1\sim_\chi\lambda_2 \overset{def}{\Longleftrightarrow} \lambda_1+\chi\in W\cdot(\lambda_2+\chi) \text{ mod } \mathfrak{h}^*\otimes\mathfrak{m}$$

Note thet every equivalence class lies in a W-orbit for a suitable ($(\rho+\chi)$-shifted) action of W on $\mathfrak{h}^*\otimes\mathbb{K}$, hence are finite.

We proved (Sec 2.3 of Lec 5 that)

$$\mathcal{O}_A=\bigoplus_{\Xi\in\Lambda/\sim_\chi}\mathcal{O}_A^{\Xi} \qquad (1)$$

where $\mathcal{O}_A^{\Xi}$ is the full subcategory of modules filtered by quotients of $\Delta(\lambda)$, $\lambda\in\Xi$.

1) Finite length

Thm: Let $A = \mathbb{K}$ be a field. Then $\forall M \in \mathcal{O}_{\mathbb{K},\chi}$ has finite length.

Fix $\Xi \in \Lambda/\sim_\chi$. For $M \in \mathcal{O}_{\mathbb{K}}^{\Xi}$ define

$$d^{\Xi}(M) := \sum_{\lambda \in \Xi} \dim_{\mathbb{K}} M_\lambda$$

- well defined b/c $\dim M_\lambda < \infty\ \forall \lambda$ & $|\Xi| < \infty$.

(1) reduces the theorem to:

Lemma: $\forall M \in \mathcal{O}_{\mathbb{K}}^{\Xi}$, the length of M is $\leqslant d^{\Xi}(M)$.

Proof:

Note that $\forall$ sub $N \subset M$, we have (by Important Exercise in Sec. 2.1 of Lec 5), $d^{\Xi}(M) = d^{\Xi}(N) + d^{\Xi}(M/N)$. It remains to show: $M \neq 0 \Rightarrow d^{\Xi}(M) > 0$. Take a non-zero homom'm $\Delta(\lambda) \xrightarrow{\varphi} M$. Then $\lambda \in \Xi$ (exercise). But since $\Delta(\lambda)_\lambda$ generates $\Delta(\lambda)$, $\varphi(\Delta(\lambda)_\lambda) \neq 0 \Rightarrow M_\lambda \neq 0 \Rightarrow$ $d^{\Xi}(M) > 0$. $\square$

2) Projective objects & projective functors

In Lec 1 we have stated several properties of projective objects in $\mathcal{O}^{pr}$ including that there are enough of them. In this lecture we start studying projectives for categories $\mathcal{O}_A$, introduced in Sec 2.1 of Lec 5 in the case when $\mathcal{O}_A$ is a local Noetherian $S(\mathfrak{h})$-algebra.

2.1) Dominant Vermas.

Definition: We say λ is χ-dominant if it's maximal in its equivalence class, Ξ, w.r.t. standard order on Λ.

Since Ξ is finite such element always exists. And if $\operatorname{im}\chi \in \mathfrak{h}^*\otimes\mathfrak{m}$, then λ is χ-dominant $\Leftrightarrow$ $\lambda+\rho$ is dominant, so χ-dominant element is unique.

Proposition: Let λ be χ-dominant. Then $\Delta_A(\lambda)$ is projective in $\mathcal{O}_A^{\Xi}$ (and hence in $\mathcal{O}_A$).

Proof:

Thanks to (1), it's enough to show this in $\mathcal{O}_A^{\Xi}$. Any object $M \in \mathcal{O}_A^{\Xi}$ is filtered by quotients of $\Delta_A(\lambda')$ for $\lambda' \in \Xi$. Any weight of $\Delta_A(\lambda')$ is $\leqslant \lambda' \Rightarrow \not> \lambda$. So the same holds for weights of $M \Rightarrow \mathfrak{n}$ kills M_λ. Hence

$$\mathrm{Hom}_{\mathcal{O}_A}(\Delta(\lambda), M) \xrightarrow{\sim} M_\lambda^{\mathfrak{n}} = M_\lambda$$

is an exact functor $\square$

Corollary. $\Delta_0(0) \in \mathcal{O}^{pr}$ is projective.

1.2) Biadjoint functors

Our tool to produce more projective objects in $\mathcal{O}_A$ is to apply functors admitting a biadjoint (both left & right adjoint). Let's recall basic compatibility between adjunction and abelian category structure:

a) Let $\mathcal{C}, \mathcal{D}$ be abelian categories & $F: \mathcal{C} \rightleftarrows \mathcal{D}: G$ be functors s.t. F is left adjoint to G. Then F is right exact & G is left exact.

b) If G is exact, then F sends projectives to projectives.

c) In particular, if G is both left & right adjoint to F, then both are exact & send projectives to projectives.

Here are examples of biadjoint functors relevant for us.

I) We have the inclusion functor $i_\Xi: \mathcal{O}_A^\Xi \hookrightarrow \mathcal{O}_A$ & its biadjoint $p_\Xi: \mathcal{O}_A \twoheadrightarrow \mathcal{O}_A^\Xi$, the projection corresponding to the direct sum decomposition (1).

II) Let V be a finite dimensional representation of $\mathfrak{g}$, $M \in \mathcal{U}(\mathfrak{g})\otimes A$-mod. We equip $V\otimes M$ with an action of $\mathcal{U}(\mathfrak{g})\otimes A$:

$a.(v\otimes m) = v\otimes am$, $x.(v\otimes m) = xv\otimes m + v\otimes xm$.

Exercise 1: Check that this indeed gives an action of $\mathcal{U}(\mathfrak{g})\otimes A$. Moreover show that:

i) if $M^0 \subset M$ generates M, then $V\otimes M^0$ generates $V\otimes M$.

ii) if $M = \bigoplus_{\lambda\in\Lambda} M_\lambda$ is a weight decomposition (so that $M_\lambda = \{m\in M \mid xm = \langle\lambda, x\rangle m + \chi(x)m \ \forall x\in\mathfrak{h}\}$), then

$V\otimes M = \bigoplus_{\lambda\in\Lambda}\left(\bigoplus_{\mu\in\Lambda} V_\mu \otimes M_{\lambda-\mu}\right)$ is the weight decomposition.

iii) if $\mathfrak{n}$ acts locally nilpotently on M, then it does so on $V\otimes M$.

Hence we get a functor $V\otimes ?: \mathcal{O}_A \longrightarrow \mathcal{O}_A$

We claim that $V^*\otimes ?$ is biadjoint to $V\otimes ?$. This is a consequence of the following form of the tensor-Hom adjunction:

Exercise 2: $\forall\, M, M' \in \mathcal{U}(\mathfrak{g})\otimes A$-mod we have the following natural isomorphism:

$$\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})\otimes A}(V\otimes M, M') \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})\otimes A}(M, V^*\otimes M')$$

Conclusion: Let Ξ, Ξ' be two equivalence classes for $\sim_X$ & V be a fin. dim. $\mathfrak{g}$-representation. We get the following functors admitting biadjoints (hence exact & sending projectives to projectives):

$$p_{\Xi'}\circ (V\otimes i_{\Xi}): \mathcal{O}_A^{\Xi} \longrightarrow \mathcal{O}_A^{\Xi'} \qquad (2)$$

these (and their direct summands as functors) are known as **projective functors**.

1.3) Behavior on Vermas.

We want to produce projective objects from a known & easy projective, a dominant Verma, by applying projective functors. So our next task is to understand the images of $\Delta_A(\lambda)$ (for arbitrary $\lambda \in \Xi$) under (2). Choose a weight basis $v_1, \dots v_n$ of V with $v_i \in V_{\mu_i}$ s.t.

$$\mu_i \leq \mu_j \text{ (in the order on } \Lambda) \Rightarrow i \geq j.$$

Exercise: $V_{\leq i} := \mathrm{Span}_{\mathbb{C}}(v_1, \dots, v_i)$ is $\mathfrak{b}$-stable.

Let $I_{\Xi'} = \{i \in \{1, \dots n\} \mid \lambda + \mu_i \in \Xi'\}$ and list the elements of $I_{\Xi'}$ as $i_1 < i_2 \dots < i_r$ ($\Rightarrow \mu_{i_j} \leq \mu_{i_k} \Rightarrow j \geq k$)

Thm: $M := p_{\Xi'}(V \otimes \Delta_A(\lambda))$ admits a filtration in $\mathcal{O}_A^{\Xi'}$
$\{0\} = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_r = M$ s.t. $M_j / M_{j-1} \simeq \Delta_A(\lambda + \mu_{i_j})$
(i.e. "smaller weights appear higher in the filtration", e.g. the quotient $M/M_{r-1} \simeq \Delta(\lambda + \mu_{i_r})$ w. minimal μ_{i_r})

Before proving the theorem we consider an example.

Example: $\mathfrak{g}=\mathfrak{sl}_2$, $\chi\in\mathfrak{h}^*\otimes\mathfrak{m}$, $\Xi=\{-1\}$, $\Xi'=\{-2,0\}$, $V=\mathbb{C}^2 \rightsquigarrow$
$M=\mathbb{C}^2\otimes\Delta_A(-1)$, $I_{\Xi'}=\{1,2\}$, $\mu_1=1$, $\mu_2=-1$. Thm predicts SES:

$$0\longrightarrow\Delta_A(0)\longrightarrow\mathbb{C}^2\otimes\Delta_A(-1)\longrightarrow\Delta_A(-2)\longrightarrow 0$$

Let's see this explicitly for $A=\mathbb{C}$ (so that we are in $\mathcal{O}_0$).
Let m_i, $i\geqslant 0$, be the weight basis of $\Delta(-1)$ w. highest vector m_0 & $m_i=f^i m_0$. Let v_1,v_2 be the usual basis of $\mathbb{C}^2$ ($ev_1=0$, $fv_1=v_2$).
We have $e(v_1\otimes m_0)=0$, $h(v_1\otimes m_0)=0 \rightsquigarrow \Delta(0)\simeq\mathcal{U}(\mathfrak{g})(v_1\otimes m_0)\subset\mathbb{C}^2\otimes\Delta(-1)$.
Next, $v_{-1}\otimes m_0$ is of weight -2 & nonzero & killed by e in the quotient yielding $\Delta(-2)\xrightarrow{\sim}(\mathbb{C}^2\otimes\Delta(-1))/\Delta(0)$.

Proof of Thm:

We first establish a similar filtration on $V\otimes\Delta_A(\lambda)$ & then apply the exact functor $p_{\Xi'}$ to this filtration.

Step 1: Note that we have the induction functor

$$\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}:\mathcal{U}(\mathfrak{b})\otimes A\text{-mod}\longrightarrow\mathcal{U}(\mathfrak{g})\otimes A\text{-mod},\ N\mapsto\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{b})}N$$

The PBW thm implies that $\mathcal{U}(\mathfrak{g})$ is a free right $\mathcal{U}(\mathfrak{b})$-module hence $\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}$ is exact. For $\lambda'\in\Lambda$ we get $\Delta_A(\lambda')=\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}A_{\lambda'}$,
where $A_{\lambda'}=A$ as an A-module w. zero action of $\mathfrak{n}$ & $\mathfrak{h}$-action by

$xa = (\langle\lambda, x\rangle + \chi(x))a.$

Step 2: We claim that $\exists$ natural iso in $\mathcal{U}(\mathfrak{g})\otimes A$-mod:

$$V\otimes \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} N \xrightarrow{\sim} \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(V\otimes N) \tag{3}$$

Indeed for $M\in \mathcal{U}(\mathfrak{g})\otimes A$-mod we have

$$\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})\otimes A}(V\otimes \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}N, M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})\otimes A}(\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}N, V^*\otimes M)$$
$$\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{U}(\mathfrak{b})\otimes A}(N, V^*\otimes M) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{U}(\mathfrak{b})\otimes A}(V\otimes N, M) \xrightarrow{\sim}$$
$$\mathrm{Hom}_{\mathcal{U}(\mathfrak{g})\otimes A}(\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(V\otimes N), M)$$ & (3) follows by Yoneda.

Step 3: So $V\otimes\Delta_A(\lambda)\simeq \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}(V\otimes A_\lambda)$. On $N = V\otimes A_\lambda$ we have a $\mathcal{U}(\mathfrak{b})\otimes A$-module filtration $N_i := V_{\leq i}\otimes A_\lambda \Rightarrow N_i/N_{i-1} \simeq \mathbb{C}_{\mu_i}\otimes A_\lambda \simeq A_{\lambda+\mu_i}$ $(i=1,..n)$. Since $\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}}$ is exact, this induces a filtration on $V\otimes\Delta_A(\lambda)$ w. successive quotients $\Delta_A(\lambda+\mu_i)$

Step 4: Apply $p_{\Xi'}$ to this filtration. It's exact & it kills $\Delta_A(\lambda+\mu_i)$ w. $\lambda+\mu_i\notin\Xi'$ & preserves $\Delta_A(\lambda+\mu_i)$ w. $\lambda+\mu_i\in\Xi'$ yielding the filtration asserted in the theorem. $\square$