RCA, PROBLEM SET 2

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0.1. Let A, B be finite dimensional associative algebras and let $\pi : A\text{-mod} \to B\text{-mod}$ be an exact functor. Prove that the following are equivalent:

- π is a quotient functor (i.e., in addition, it admits a right inverse),
- there is a projective A-module P together with an isomorphism $\operatorname{End}_A(P)^{opp} \xrightarrow{\sim} B$ such that $\pi = \operatorname{Hom}_A(P, \bullet)$.

0.2. Let A, B be finite dimensional algebras and $\iota : A \to B$ is an algebra homomorphism. Then the homomorphism ι is surjective if and only if any A-submodule of a B-module is B-stable.

0.3. Check that an H_c-module M lies in $\mathcal{O}$ if and only if $M = \bigoplus_\lambda M_\lambda$, all M_λ are finite dimensional, and the set $\{\lambda | M_\lambda \neq 0\}$ is bounded from above, in the sense, for example, that the real parts are bounded. Also check that, for $M \in \mathcal{O}_c(W, \mathfrak{h}^*)$, we have $(M^\vee)_\lambda = (M_\lambda)^*$ (where $\bullet^\vee$ is the duality introduced in the lecture). Deduce that $M^\vee \in \mathcal{O}_c(W, \mathfrak{h})$.

0.4. Prove that the $\nabla_c(\tau)$'s are the costandard objects in the highest weight category $\mathcal{O}_c$ meaning, for example, that $\dim \operatorname{Hom}(\Delta_c(\tau), \nabla_c(\tau')) = \delta_{\tau,\tau'}$ and $\operatorname{Ext}^1_{\mathcal{O}_c}(\Delta_c(\tau), \nabla_c(\tau')) = 0$ for all τ, τ'.

0.5. Prove the following faithfulness properties of KZ.

a) Show that $\Delta_c(\tau)$ has no nonzero subobjects killed by KZ. Deduce that KZ is faithful on the standardly filtered objects.

b) Show that $\nabla_c(\tau)$ has no nonzero subquotients killed by KZ. Deduce that KZ is faithful on the costandardly filtered objects.

c) Deduce that KZ is fully faithful on the tilting objects (i.e., the objects that are both standardly filtered and costandardly filtered).

0.6. Let c be integer valued. Then $\mathcal{O}_c(W)$ is semisimple.

0.7. Extra credit... That's how GGOR proved that KZ is fully faithful on the projectives.

a) Consider the functor $D : D^b(H_c\text{-mod}) \to D^b(H_c^{opp}\text{-mod}), M \mapsto R\operatorname{Hom}(M, H_c)[\dim \mathfrak{h}]$. Show that it maps $\Delta_c(\tau)$ to $\Delta_c^{opp}(\tau \otimes \mathrm{sgn})$, where the superscript indicates the right handed analog of the Verma module.

b) Show that D descends to an equivalence $D^b(\mathcal{O}_c) \xrightarrow{\sim} D^b(\mathcal{O}_c^{opp})$ that respects the subcategories of torsion objects and is t-exact on the quotients.

c) Show that D is a contravariant Ringel duality (to learn about those you could look at the highest weight problem set). Deduce from (c) of Problem 5 that the quotient functor $\mathcal{O}_c \twoheadrightarrow \mathcal{O}_c/\mathcal{O}_c^{tor}$ is fully faithful on the projective objects.