# LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

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## 1. Kleinian singularities

Kleinian singularities are remarkable singular affine surfaces (varieties of dimension 2). They arise as quotients of $\mathbb{C}^2$ by finite subgroups of $\mathrm{SL}_2(\mathbb{C})$. Our main interest is not in these singularities themselves but in their (not necessarily commutative) deformations.

This lecture is organized as follows. First, 1.1, we present Kleinian singularities as surfaces in $\mathbb{C}^3$. Next, we recall the classification of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ in 1.2. In 1.3 we present the simplest version of the so called *McKay correspondence* that relates finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ to Dynkin diagrams. Then we relate the singularities and the subgroups as promised in the previous paragraph. One advantage of this realization, is that the singularities acquire a natural *grading*. We discuss graded algebras in 1.5.

After that we start to proceed to our next topic and discuss the general notion of a deformation. Finally, we sketch a purely algebro-geometric way to connect the Kleinian singularities to Dynkin diagrams, 1.7.

For more information on Kleinian singularities (and, in particular, their relation to simple Lie algebras) see [Sl], Section 6, in particular.

1.1. **Singularities.** There are some remarkable singular affine algebraic varieties of dimension 2. They have many names (Kleinian singularities, rational double points, du Val singularities) and also many nice properties (e.g., these are only normal Gorenstein singularities in dimension 2). They can be described very explicitly, as surfaces in $\mathbb{C}^3$ given by a single equation on the variables $x_1, x_2, x_3$. They split into two families and three exceptional types. Here are the equations

$(A_r)$ $x_1^{r+1} + x_2x_3 = 0$, $r \geqslant 1$.
$(D_r)$ $x_1^{r-1} + x_1x_2^2 + x_3^2$, $r \geqslant 4$.
$(E_6)$ $x_1^4 + x_2^3 + x_3^2 = 0$.
$(E_7)$ $x_1^3x_2 + x_2^3 + x_3^2 = 0$.
$(E_8)$ $x_1^5 + x_2^3 + x_3^2 = 0$.

Of course, $A_r, D_r, E_6, E_7, E_8$ are precisely the simply laced Dynkin diagrams. In a way, this and three subsequent lectures are to explain relationship between the singularities and the diagrams.

1.2. **Finite subgroups of** $\mathrm{SL}_2(\mathbb{C})$**.** It turns out that the Kleinian singularities can be realized as quotients of $\mathbb{C}^2$ by finite subgroups of $\mathrm{SL}_2(\mathbb{C})$. Let us recall their classification.

Any finite subgroup in $\mathrm{SL}_2(\mathbb{C})$ admits an invariant hermitian product on $\mathbb{C}^2$ and so is conjugate to a subgroup in $\mathrm{SU}_2$. Recall that there is a covering $\mathrm{SU}_2 \twoheadrightarrow \mathrm{SU}_2/\{\pm E\} \cong \mathrm{SO}_3(\mathbb{R})$ given by the adjoint representation of $\mathrm{SU}_2$. So the first step in classifying finite subgroups of $\mathrm{SU}_2$ is to classify those in $\mathrm{SO}_3(\mathbb{R})$.

Inside $\mathrm{SO}_3(\mathbb{R})$ we have the following finite subgroups:

(1) The cyclic group of order $n$, its generator is a rotation by the angle of $2\pi/n$.

(2) The dihedral group of order $2n$ with $n \geqslant 2$ realized as the group of rotation symmetries of a regular $n$-gon on the plane inside of the 3D space. Of course, a regular 2-gon is just a segment.
(3) The group of rotational symmetries of the regular tetrahedron isomorphic to the alternating group $A_4$.
(4) The group of rotational symmetries of the regular cube/octahedron isomorphic to the symmetric group $S_4$.
(5) The group of rotational symmetries of the regular dodecahedron/icosahedron isomorphic to $A_5$.

**Problem 1.1.** *Prove that these group form a complete list of finite subgroups of* $\mathrm{SO}_3(\mathbb{R})$. *You may use the following strategy. Let $G$ be a finite subgroup of* $\mathrm{SO}_3(\mathbb{R})$. *Consider its action on the unit sphere. Show that any non-unit element of $G$ fixes a unique pair of opposite points and that the stabilizer of each point $P$ is cyclic of some order, say, $n_P$. Choose representatives $P_1, \ldots, P_k$ of orbits with non-trivial stabilizers, one in each orbit. Show that*

$$2\left(1-\frac{1}{n}\right)=\sum_{i=1}^{k}\left(1-\frac{1}{n_{P_i}}\right).$$

*Analyze the possibilities for the numbers $n, n_{P_1}, \ldots, n_{P_k}$ and deduce the classification.*

Now the classification of finite subgroups in $\mathrm{SL}_2(\mathbb{C})$ up to conjugacy is as follows.

($A_r$) The cyclic group of order $r+1$, i.e., $\{\begin{pmatrix}\epsilon & 0\\ 0 & \epsilon^{-1}\end{pmatrix} | \epsilon^{r+1}=1\}$.
($D_r$) The dihedral group of order $4(r-2)$ with $r \geqslant 4$, i.e., $\{\begin{pmatrix}\epsilon & 0\\ 0 & \epsilon^{-1}\end{pmatrix}, \begin{pmatrix}0 & \epsilon\\ -\epsilon^{-1} & 0\end{pmatrix} | \epsilon^{2(r-2)} = 1\}$.
($E_6$) The double cover of $A_4 \subset \mathrm{SO}_3(\mathbb{R})$.
($E_7$) The double cover of $S_4 \subset \mathrm{SO}_3(\mathbb{R})$.
($E_8$) The double cover of $A_5 \subset \mathrm{SO}_3(\mathbb{R})$.

**Problem 1.2.** *Use the result of Problem 1.1 to deduce this classification.*

1.3. **McKay correspondence I.** Again, we have labeled the finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ by simply laced Dynkin diagrams. To get a diagram from a subgroup one can use the recipe called the *McKay correspondence* to be described now.

Pick a finite subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$. Let $N_0, \ldots, N_r$ denote the irreducible representations of $\Gamma$ and suppose that $N_0$ is the trivial representation. Set $m_{ij} = \dim \mathrm{Hom}_\Gamma(N_i \otimes \mathbb{C}^2, N_j)$, where $\mathbb{C}^2$ denotes the representation of $\Gamma$ coming from the inclusion $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$. The representation $\mathbb{C}^2$ is self-dual because it admits an invariant non-degenerate skew-symmetric bilinear form. It follows that $m_{ij} = \dim \mathrm{Hom}_\Gamma(N_i \otimes \mathbb{C}^2, N_j) = \dim \mathrm{Hom}_\Gamma(N_i, N_j \otimes \mathbb{C}^2) = \dim \mathrm{Hom}_\Gamma(N_j \otimes \mathbb{C}^2, N_i) = m_{ji}$.

Consider the graph with vertices $0, \ldots, r$ and the number of edges between $i$ and $j$ equal to $m_{ij}$. It is called the *McKay graph* of $\Gamma$.

To state the result we need to recall various things regarding root systems.

Fix a simply laced Dynkin diagram with vertices labeled by $1, \ldots, r$. To the diagram one assigns a root system in the Euclidian space $\mathbb{R}^n$ together with its subset, a simple root system, say $\alpha_1, \ldots, \alpha_r$, that constitute a basis in $\mathbb{R}^n$. The vertices $i, j$ are connected if and only if $(\alpha_i, \alpha_j) \neq 0$ (in which case one necessarily has $(\alpha_i, \alpha_j) = -1$ – we only consider simply laced diagrams). In the root system, there is unique maximal root, say $\delta$, that can be, in the simply laced case, characterized by the property that $(\alpha_i, \delta) \geqslant 0$ for all $i$. Set $\alpha_0 := -\delta$. Then the *extended Dynkin diagram*, by definition, has vertices $0, \ldots, r$ and the vertices are connected according to the same rule as for the usual Dynkin diagram.

The following result is the most elementary form of the so called McKay correspondence.

**Proposition 1.1.** *The McKay graph $\Gamma$ is the extended Dynkin diagram corresponding to the finite Dynkin diagram labeling $\Gamma$. Moreover, 0 is the extending vertex. Finally, one has $\sum_{i=0}^{r} \dim N_i \cdot \alpha_i = 0$.*

We are not going to prove Proposition 1.1 in all cases. We are going to consider the easiest case as an example and propose two more cases in the form of problems.

The example we are going to consider is, of course, the cyclic group, $\Gamma = \{\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} | \epsilon^{r+1} = 1\}$. Since the group is abelian, all irreducible representations are 1-dimensional and are given by characters. Let $N_i$, denote the representation corresponding to the character given by $\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \mapsto \epsilon^i$ so that $N_i = N_j$ if and only if $i-j$ is divisible by $r+1$. Of course, $\mathbb{C}^2 = N_{-1} \oplus N_1$. So $\dim \mathrm{Hom}_\Gamma(N_i \otimes \mathbb{C}^2, N_j) = 1$ if $i - j = \pm 1 \mod r+1$ and $\dim \mathrm{Hom}_\Gamma(N_i \otimes \mathbb{C}^2, N_j) = 0$, else. Therefore the McKay graph is cyclic with $r+1$ vertices.

On the other hand, the corresponding root system can be realized in the space $\{(x_1, \ldots, x_{r+1}) \in \mathbb{R}^{r+1} | \sum_{i=1}^{r+1} x_i = 0\}$ with the scalar product restricted from the standard one on $\mathbb{R}^{r+1}$. The simple roots are $\alpha_i = e_i - e_{i+1}, i = 1, \ldots, r$, where $e_1, \ldots, e_{r+1}$ denote the tautological basis elements in $\mathbb{R}^{r+1}$. The whole root system consists of the elements of the form $\{e_i - e_j, i \neq j\}$. Further, $\alpha_0 = e_{r+1} - e_1$. From here we see that the extended Dynkin diagram is again the cyclic graph with $r+1$ vertices, and $\sum_{i=0}^{r} \alpha_i = 0$.

**Problem 1.3.** *Check Proposition 1.1 for the group of type $D_r$.*

**Problem 1.4.** *This problem discusses the group of type $E_6$.*

*1) We start with a construction. Take the group $Q_8$ of unit quaternions. It has elements $\{\pm 1, \pm i, \pm j, \pm k\}$. Show that the cyclic group $\mathbb{Z}_3$ acts on $Q_8$ by automorphisms in such a way that the generator $\omega$ acts as follows: $\omega(-1) = -1, \omega(i) = j, \omega(j) = k, \omega(k) = i$. Embed the semi-direct product $\Gamma := \mathbb{Z}_3 \ltimes Q_8$ into $\mathrm{SL}_2(\mathbb{C})$. Further, show that $\Gamma/\{\pm 1\} \cong A_4$.*

*2) Show that $\Gamma$ has 3 one-dimensional, 3 two-dimensional and 1 three-dimensional irreducible representations.*

*3) Prove Proposition 1.1 in this case.*

1.4. **Quotients.** Our goal here is to show that the Kleinian singularities are actually quotients of $\mathbb{C}^2$ by the action of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$.

First, let us recall some general properties of quotients under finite group actions. Let $X$ be an affine algebraic variety and $\Gamma$ be a group acting on $X$ by automorphisms. Then $\Gamma$ also acts on the algebra $\mathbb{C}[X]$ of regular functions on $X$: for $g \in \Gamma, f \in \mathbb{C}[X]$ and $x \in X$ we have $g.f(x) := f(g^{-1}x)$. The $\Gamma$-invariant elements in $\mathbb{C}[X]$ form a subalgebra denoted by $\mathbb{C}[X]^\Gamma$.

Now suppose that $\Gamma$ is finite. We will need the following three classical results from Invariant theory, see [PV],[Sp] for references.

A) The algebra $\mathbb{C}[X]^\Gamma$ is finitely generated. In particular, we can form the corresponding algebraic variety to be denoted by $X/\Gamma$.

B) The inclusion $\mathbb{C}[X]^\Gamma \subset \mathbb{C}[X]$ gives rise to a morphism $\pi : X \to X/\Gamma$ of algebraic varieties called the quotient morphism. Each fiber of this morphism is a single $\Gamma$-orbit. This means that the variety $X/\Gamma$ parameterizes $\Gamma$-orbits on $X$.

C) The algebra $\mathbb{C}[X]$ is finite over $\mathbb{C}[X]^\Gamma$. Equivalently, the morphism $\pi$ is finite.

Also it is easy to see that $X/\Gamma$ is irreducible provided $X$ is. Property C) implies that the dimensions of $X$ and $X/\Gamma$ coincide.

Let us return to the case of finite subgroups of $\mathrm{SL}_2(\mathbb{C})$. The group $\mathrm{SL}_2(\mathbb{C})$ acts naturally on $\mathbb{C}^2$. Let $\Gamma$ be a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$.

**Proposition 1.2.** *The variety $\mathbb{C}^2/\Gamma$ is isomorphic to the Kleinian singularity of the same type as $\Gamma$.*

Again, let us show this in the simplest example of a cyclic group. The algebra of regular functions on $\mathbb{C}^2$ is nothing else but the polynomial algebra $\mathbb{C}[x,y]$. A generator $g = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{pmatrix}$ acts on a monomial $x^n y^m$ as follows: $g.x^n y^m = \epsilon^{m-n} x^n y^m$. From here we see that the invariant subalgebra $\mathbb{C}[x,y]^\Gamma$ is spanned by all monomials $x^n y^m$, where $n-m$ is divisible by $r+1$. Any such monomial can be written in the form $x_1^a x_2^b x_3^c$, where $x_1 := xy, x_2 := x^{r+1}, x_3 := y^{r+1}$. So $\mathbb{C}[x,y]^\Gamma$ is generated by $x_1, x_2, x_3$. So $\mathbb{C}^2/\Gamma$ is realized as an (automatically, irreducible) subvariety in $\mathbb{C}^3$. On the other hand, by C), the dimension of $\mathbb{C}^2/\Gamma$ equals 2. Therefore $\mathbb{C}^2/\Gamma$ is a divisor and hence can be defined by a single equation. Clearly, the elements $x_1, x_2, x_3$ satisfy the relation $x_1^{r+1} = x_2 x_3$. The polynomial $x_2x_3 - x_1^{r+1}$ is easily seen to be irreducible. So $\mathbb{C}[x,y]^\Gamma = \mathbb{C}[x_1,x_2,x_3]/(x_1^{r+1} - x_2x_3)$, and we are done.

**Problem 1.5.** *Prove Proposition 1.2 for the dihedral groups.*

In the sequel we will basically need only a quotient realization of Kleinian singularities.

1.5. **Graded algebras.** Our goal here is to introduce a natural grading on $\mathbb{C}[x,y]^\Gamma$.

We start with a general definition of a graded algebra. Let $A$ be an associative algebra with unit. Let $\mathbb{Z}_{\geqslant 0}$ denote the set of non-negative integers. A $\mathbb{Z}_{\geqslant 0}$-grading on $A$ is a decomposition $A = \bigoplus_{n \geqslant 0} A^n$ into the direct sum of subspaces subject to the following conditions:

- $A^n A^m \subset A^{n+m}$.
- $1 \in A^0$.

In the case when $A^0$ is spanned by 1, we will say that the grading is positive. Analogously, one can define the notions of a $\mathbb{Z}$-grading, $\mathbb{Z}^n$-grading, etc. Of course, by a graded algebra one means an algebra equipped with a grading. Informally, the definition of a graded algebra is given in such a way that we have a notion of a "homogeneous element of degree $n$".

For example, consider the polynomial algebra $A := \mathbb{C}[x_1, \ldots, x_n]$. It is positively graded, the space $A^m$, by definition, is spanned by all monomials of degree $m$. Another example is the tensor algebra $T(V)$ of a vector space $V$. By definition, $T(V) = \bigoplus_{n=0}^\infty V^{\otimes n}$ and we set $T(V)^n := V^{\otimes n}$. With a chosen basis in $V$, say $x_1, \ldots, x_n$, we can identify $T(V)$ with the free algebra $\mathbb{C}\langle x_1, \ldots, x_n \rangle$ generated by $x_1, \ldots, x_n$. For a basis in $\mathbb{C}\langle x_1, \ldots, x_n \rangle$ we can take all noncommutative monomials in the elements $x_i$.

More examples of graded algebras can be obtained by taking quotients of graded algebras by homogeneous ideals. By definition, a subspace $I$ in a graded algebra is called *homogeneous* if $I = \bigoplus_n I^n$, where $I^n := A^n \cap I$. Then, of course, $A/I = \bigoplus_n A^n/I^n$ and we set $(A/I)^n := A^n/I^n$. It is straightforward to check that if $I$ is, in addition, a two-sided ideal, then $A/I = \bigoplus_n (A/I)^n$ is a grading. Of course, a two-sided ideal generated by homogeneous elements is homogeneous.

For example, $\mathbb{C}[x_1, \ldots, x_n]$ is the quotient of $\mathbb{C}\langle x_1, \ldots, x_n \rangle$ by the relations $x_ix_j - x_jx_i = 0$. Another example is provided by the exterior algebra $\Lambda(V)$ that is the quotient of $T(V)$ by the relations $u \otimes v + v \otimes u = 0, u, v \in V$.

Or we can realize graded algebras as subalgebras. Let $A = \bigoplus_{n\geqslant 0} A^n$ be a graded algebra and $B$ be its subalgebra. We say that $B$ is a graded subalgebra if $B = \bigoplus_{n\geqslant 0} B^n$, where $B^n := A^n \cap B$. Then, of course, the decomposition $B = \bigoplus_{n\geqslant 0} B^n$ is a grading. For example, take $A = \mathbb{C}[x, y]$ and $B := \mathbb{C}[x, y]^\Gamma$. Since $\Gamma$ preserves the graded components of $A$, the subalgebra $B$ is graded. In more elementary terms, an element in $B$ is homogeneous of degree $n$ if it is so in $A$.

1.6. **Definition of deformation.** In this course, we are not much interested in quotient singularities themselves (or their algebras of functions). Rather we want to study their both commutative and non-commutative deformations. In this lecture we start our discussion of deformations giving a general algebraic definition.

Let $B$ be a commutative algebra and let $\mathfrak{m}$ be a maximal ideal of $B$. Further, let $A$ be an associative algebra such that $B$ is embedded into the center of $A$.

**Definition 1.3.** Let $A_0$ be an associative algebra. We say that $A$ is a deformation of $A_0$ over $B$ if $A_0 \cong A/A\mathfrak{m}$ and $A$ is flat as a $B$-module.

Recall that a $B$-module $M$ is called *flat* if the functor $M \otimes_B \bullet$ is exact (in general, this functor is only right exact). For example, if $B = \mathbb{C}[x]$, the polynomial algebra in one variable, then $M$ is flat if and only if the polynomial $x - \alpha$ is a not a zero divisor in $M$ for ant $\alpha \in \mathbb{C}$.

If the algebra $B$ in the previous definition is the algebra $\mathbb{C}[X]$ of regular functions on some affine variety, then to any point $x \in X$ we can assign the quotient $A_x := A/A\mathfrak{m}_x$, where $\mathfrak{m}_x$ is the maximal ideal of $x$ in $\mathbb{C}[X]$. So we get a family $A_x$ of associative algebras parameterized by points of $X$. The flatness assumption informally means that $A_x$ depends continuously on $x$.

For example, let $B = \mathbb{C}[c_0, \ldots, c_{r-1}]$. Then we can set $A = B[x_1, x_2, x_3]/(x_2x_3 - x_1^{r+1} - \sum_{i=0}^{r-1} c_i x_1^i)$. If we take $\mathfrak{m} = (c_0, \ldots, c_{r-1})$, then we see that $A_0 = \mathbb{C}[x_1, x_2, x_3]/(x_2x_3 - x_1^{r+1})$ – the singularity of type $A_r$. We will not check flatness – usually to do this is not trivial. But modulo the flatness condition, $A$ is a deformation of $A_0$.

1.7. **Resolutions.** Let us explain an algebro-geometric way to attach a Dynkin diagram to a singularity $X$ from our list. Namely, recall that by a resolution of $X$ one means a smooth (but non-affine) variety $\tilde{X}$ equipped with a projective birational morphism $\tilde{X} \to X$. Since we are in dimension 2, there is a *minimal* resolution $\pi : \tilde{X}_{min} \to X$, the minimality condition means that any other resolution factors through $X$.

The point 0 in $\mathbb{C}^3$ belongs to $X$. The fiber $\pi^{-1}(0)$ happens to be a union of projective lines, say $P_1, \ldots, P_r$. It turns out that the intersection matrix of the $P_i$'s is the negative of the Cartan matrix of a Dynkin diagram, and this is precisely the diagram of the type of a singularity.

## References

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