LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

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10. Moment maps in algebraic setting

10.1. **Symplectic algebraic varieties.** An affine algebraic variety X is said to be *Poisson* if $\mathbb{C}[X]$ is equipped with a Poisson bracket.

Exercise 10.1. *Let A be a commutative algebra and B be a localization of A. Let A be equipped with a bracket. Show that there is a unique bracket on B such that the natural homomorphism $A \to B$ respects the bracket.*

Thanks to this exercise, the *sheaf* $\mathcal{O}_X$ of regular functions on X acquires a bracket (i.e., we have brackets on all algebras of sections and the restriction homomorphisms are compatible with the bracket). We say that an arbitrary (=not necessarily affine) variety X is Poisson if the sheaf $\mathcal{O}_X$ comes equipped with a Poisson bracket.

Recall that on a variety X such that $\mathcal{O}_X$ is equipped with a bracket we have a bivector (=a bivector field) $P \in \Gamma(X^{reg}, \bigwedge^2 TX^{reg})$. This gives rise to a map $v_x : T^*_x X \to T_x X$ for $x \in X^{reg}, \alpha \mapsto P_x(\alpha, \cdot)$. We say that P is nondegenerate in x if this map is an isomorphism. In this case, we can use this map to get a 2-form $\omega_x \in \bigwedge^2 T^*_x X$: $\omega_x(v_x(\alpha), v_x(\beta)) = P_x(\alpha, \beta) = \langle \alpha, v_x(\beta)\rangle = -\langle v_x(\alpha), \beta\rangle$.

Now suppose X is smooth. Suppose that P is non-degenerate (=non-degenerate at all points). So we have a non-degenerate form ω on X. The condition that P is Poisson is equivalent to $d\omega = 0$. A non-degenerate closed form ω is called *symplectic* (and X is called a *symplectic variety*).

The most important for us class of symplectic varieties is cotangent bundles. Let X_0 be a smooth algebraic variety, set $X := T^*X_0$. A symplectic form ω on X is introduced as follows. First, let us introduce a canonical 1-form α. We need to say how α_x pairs with a tangent vector for any $x \in X$. A point X can be thought as a pair (x_0, β), where $x_0 \in X_0$ and $\beta \in T^*_{x_0} X_0$. Consider the projection $\pi : X \twoheadrightarrow X_0$ (defined by $\pi(x) = x_0$). For $x = (x_0, \beta)$ we define α_x by $\langle \alpha_x, v\rangle = \langle \beta, d_x\pi(v)\rangle$.

We can write α in "coordinates". If we worked in the C^∞- or analytic setting, we could use the usual coordinates. However, we cannot do this because we want to show that α is an algebraic form. So we will use an algebro-geometric substitute for coordinate charts: étale coordinates. Namely, we can introduce étale coordinates in a neighborhood of each point $x_0 \in X_0$. Let us choose functions $x^1, \ldots, x^n$ with a property that $d_{x_0}x^1, \ldots d_{x_0}x^n$ form a basis in $T^*_{x_0}X_0$. Then $dx^1, \ldots, dx^n$ are linearly independent at any point from some neighborhood X_0^0 of x_0. So the map $\varphi : X_0^0 \to \mathbb{C}^n$ given by $(x^1, \ldots, x^n)$ is étale and we call $x^1, \ldots, x^n$ étale coordinates. Then we can get étale coordinates $y_1, \ldots, y_n$ on $T^*X_0^0$ as follows: by definition $y^i(x_0, \beta)$ is the coefficient of $d_{x_0}x^i$ in β, i.e., $\beta = \sum_{i=1}^n y_i(x_0, \beta) d_{x_0}x^i$ (and we view $x^1, \ldots, x^n$ as functions on $T^*X_0^0$ via pull-back). Then, on $T^*X_0^0$, α is given by $\sum_{i=1}^n y_i dx^i$.

There is an important remark about α: it is canonical. In particular, if we have a group action on X_0, it naturally lifts to T^*X_0: $g(x_0, \beta) = (gx_0, g_{*x_0}\beta)$, where g_{*x_0} is the isomorphism

$T^*_{x_0}X_0 \to T^*_{gx_0}X_0$ induced by g. The coordinate free definition of α implies that α is invariant under any such group action on T^*X_0.

Now set $\omega = -d\alpha$ so that, in the étale coordinates, $\omega = \sum_{i=1}^n dx^i \wedge dy_i$. We immediately see that ω is a symplectic form. Also let us point out that if X_0 is a vector space, then ω is a constant form (=skew-symmetric bilinear form) on the double vector space $X_0 \oplus X_0^*$. The remark in the previous paragraph applies to ω as well.

10.2. **Hamiltonian vector fields.** Let X be a Poisson variety and f be a local section of $\mathcal{O}_X$. Then we can form the vector field $v(f) = P(df, \cdot)$ (defined in the domain of definition of f). This is called the *Hamiltonian vector field* (or the *skew gradient*) of f. Clearly, v is linear, and satisfies the Leibniz identity $v(fg) = gv(f) + fv(g)$. Further, we have

$$L_{v(f)}g = -\langle v(f), dg\rangle = \{f, g\}. \tag{1}$$

Here and below we write L_ξ for the Lie derivative of ξ so that $L_\xi f = -\partial_\xi f$. Recall that in the C^∞-situation, the Lie derivative is defined as follows. We pick a flow $g(t)$ produced by the vector field ξ and then for a tensor field τ define $L_\xi\tau = \frac{d}{dt}g(t)_*\tau|_{t=0}$. In particular, if τ is a function f, then we get $L_\xi(f) = \frac{d}{dt}f(g(-t))|_{t=0} = -\partial_\xi f$. If τ is a vector field, then $L_\xi\tau = [\xi, \tau]$, where, by convention, the bracket on the vector fields is introduced by $L_{[\xi,\eta]}f = [L_\xi, L_\eta]f$. Finally, if τ is a form, then we have the Cartan formula:

$$L_\xi\tau = -d\iota_\xi\tau - \iota_\xi d\tau, \tag{2}$$

where ι_ξ stands for the contraction with ξ (as the first argument): $\iota_\xi\tau(\ldots) = \tau(\xi, \ldots)$. In particular, if both ξ and τ are algebraic, then so is $L_\xi\tau$, and we can define $L_\xi\tau$ using the formulas above.

Using (1) and the Jacobi identity for $\{\cdot,\cdot\}$, we deduce that the map $f \mapsto v(f)$ is a Lie algebra homomorphism. Also we remark that every Hamiltonian vector field is Poisson, i.e.,

$$L_{v(f)}P = 0 \tag{3}$$

(this is yet another way to state the Jaconi identity for $\{\cdot,\cdot\}$).

If X is symplectic, we can rewrite the definition of the Hamiltonian vector field as

$$\iota_{v(f)}\omega = df. \tag{4}$$

Also we have

$$\omega(v(f), v(g)) = \{f, g\} \tag{5}$$

and

$$L_{v(f)}\omega = 0. \tag{6}$$

So in this case $f \mapsto v(f)$ is a Lie algebra homomorphism between $\mathbb{C}[X]$ and the algebra $\mathrm{SVect}(X)$ of symplectic vector fields on X.

Consider the case of $X = T^*X_0$, where, for simplicity, we assume that X_0 is affine. Then $\mathbb{C}[X] = S_{\mathbb{C}[X_0]}(\mathrm{Vect}(X_0))$. As a function on $\mathbb{C}[X]$ the vector field ξ is given by

$$\xi(x_0, \beta) = \langle \beta, \xi_{x_0}\rangle. \tag{7}$$

Let us compute the vector fields $v(f), f \in \mathbb{C}[X_0]$, and $v(\xi), \xi \in \mathrm{Vect}(X_0)$. We claim that $v(f) = -df$, viewed as a vertical vector field on T^*X_0, its value on the fiber $T^*_{x_0}X_0$ is constant $-d_{x_0}f$. To avoid confusion below we will write Df for the vector field df. Further, to a vector field ξ on X_0 we can assign a vector field $\tilde{\xi}$ on X by requiring $L_{\tilde{\xi}}\eta = [\xi, \eta], L_{\tilde{\xi}}g = L_\xi g$ for $\eta \in \mathrm{Vect}(X_0), g \in \mathbb{C}[X_0]$. We claim that $v(\xi) = \tilde{\xi}$. The vector field $\tilde{\xi}$ has the following meaning. Assume that we are in the C^∞-setting. Then to ξ we can assign its flow $g(t)$ (of diffeomorphisms of X_0). Then we can canonically lift this flow to T^*X_0. The vector field $\tilde{\xi}$ is associated to the lifted flow. In particular, from this description one sees that $d_{(x_0,\beta)}\tilde{\xi} = \xi_{x_0}$.

Applying (2) to $\tau = \alpha$ and a vector field η on T^*X_0, and using $-d\alpha = \omega$, we get $L_\eta\alpha = -d\iota_\eta\alpha + \iota_\eta\omega$ and so

$$\iota_\eta\omega = L_\eta\alpha + \iota_\eta\omega. \quad (8)$$

If $\eta = -Df$, then $\iota_\eta\alpha = 0$ (α vanishes on all vertical vector fields by the coordinate free construction). So we get $\iota_{-Df}\omega = L_{-Df}\alpha$. Again, the construction of α implies that $L_{-Df}\alpha = \partial_{Df}\alpha = df$ (in local coordinates we have $\partial_{Df}\alpha = \sum_{i=1}^n \partial_{Df}y_i dx^i = \sum_{i=1}^n \partial_{x_i} f dx^i = df$). So $\iota_{-Df}\omega = df = \iota_{v(f)}\omega$ so $v(f) = -Df$.

Now let us check that $v(\xi) = \tilde{\xi}$. We claim that $L_{\tilde{\xi}}\alpha = 0$. In the C^∞-setting, this follows from the observation that α is preserved by any diffeomorphism of T^*X_0 lifted from X_0. Since all formulas in the algebraic setting are the same as in the C^∞ one, we get our claim. Also we remark that by the construction of $\tilde{\xi}$, we have $d\pi(\tilde{\xi}) = \xi$ and therefore, thanks to (7), $\iota_{\tilde{\xi}}\alpha = \xi$ (as functions on T^*X_0). So we have $\iota_{\tilde{\xi}}\omega = d\iota_{\tilde{\xi}}\alpha$. But $\iota_{\tilde{\xi}}\alpha$ is ξ, by the description of the function ξ above.

Exercise 10.2. *Show that the Poisson bracket on $\mathbb{C}[X]$ can be characterized as follows: we have $\{f, g\} = 0, \{\xi, f\} = L_\xi f, \{\xi, \eta\} = [\xi, \eta]$ for $f, g \in \mathbb{C}[X_0], \xi, \eta \in \mathrm{Vect}(X_0)$. Deduce that, with respect to the standard grading on $\mathbb{C}[X] = S_{\mathbb{C}[X_0]}(\mathrm{Vect}(X_0))$, the bracket has degree -1.*

The construction of Hamiltonian vector fields is of importance in Classical Mechanics. Namely, we can consider a mechanical system on a Poisson variety X whose velocity vector is $-v(H)$ so that $\frac{d}{dt}f(x(t)) = (L_{v(H)}f)(x(t))$. In this case, the function H is interpreted as the *Hamiltonian* (=the full, i.e., "kinetic + potential", energy) of this system. The condition on a function F to be a first integral (=preserved quantity) of this system is $L_{v(H)}F = 0$, i.e., $\{H, F\} = 0$. In particular, H itself is the first integral (the energy conservation law).

Let us consider a very classical example. Let X_0 (a configuration space) be an open subset in $\mathbb{C}^n$ with coordinates $x^1, \ldots, x^n$. Consider the mechanical system with potential $V = V(x^1, \ldots, x^n)$, its evolution is given by $\ddot{x}^i = -\frac{\partial V}{\partial x^i}$. Introduce new variables $y_i = \dot{x}^i$ and the Hamiltonian $H = \frac{1}{2}\sum_{i=1}^n y_i^2 + V$. Then we can rewrite the system as $\dot{x}^i = y_i = \frac{\partial H}{\partial y_i} = -\{H, x_i\}, \dot{y}_i = -\frac{\partial H}{\partial x^i} = -\{H, y_i\}$. So H becomes the Hamiltonian of our system (considered on the *phase space* T^*X_0).

10.3. **Moment maps.** Now let X be a smooth variety equipped with an action of an algebraic group G. Let $\mathfrak{g}$ be the Lie algebra of G. To the G-action one assigns a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{Vect}(X), \xi \mapsto \xi_X$. In the C^∞-setting, ξ_X is the vector field associated to the flow $\exp(t\xi)$. The definition in the algebraic setting is a bit more subtle. If X is affine, then this homomorphism can be described as follows. We have the induced action of G on $\mathbb{C}[X]$. Every function lies in a finite dimensional G-stable subspace. So we have a representation of $\mathfrak{g}$ in $\mathbb{C}[X]$ and this representation is by derivations, let ξ_X be the derivation corresponding to ξ. An important special case: if X is a vector space and the G-action is linear, then $\xi_{X,x} = \xi x$, the image of x under the operator corresponding to ξ. We write ξ_x for the value of this field at $x \in X$, and $\mathfrak{g}x$ for $\{\xi_x | \xi \in \mathfrak{g}\}$, of course, $\mathfrak{g}x = T_x(Gx)$. In the non-affine case one needs to use some structural results regarding algebraic group actions.

Now assume that X is symplectic with form ω and that G preserves ω. Then $L_{\xi_X}\omega = 0$ so we have a homomorphism $\mathfrak{g} \to \mathrm{SVect}(X)$. This homomorphism is obviously G-equivariant.

Also we have a homomorphism $\mathbb{C}[X] \to \mathrm{SVect}(X)$ given by $f \mapsto v(f)$, it is also G-equivariant. We say that the action is *Hamiltonian*, if there is a G-equivariant Lie algebra homomorphism $\xi \mapsto H_\xi, \mathfrak{g} \to \mathbb{C}[X]$ such that $v(H_\xi) = \xi_X$.

Exercise 10.3. *Show that a G-equivariant map $\xi \mapsto H_\xi$ with $v(H_\xi) = \xi_X$ is automatically a Lie algebra homomorphism.*

The map $\xi \mapsto H_\xi$ is called *a comoment map*. By the *moment map* we mean the dual map, $\mu : X \to \mathfrak{g}^*$, given by $\langle \mu(x), \xi \rangle = H_\xi(x)$ (this map is dual to the homomorphism $\mathbb{C}[\mathfrak{g}^*] = S(\mathfrak{g}) \to \mathbb{C}[X]$ induced by $\xi \mapsto H_\xi$). The map μ is G-equivariant and satisfies $\langle d_x\mu, \xi \rangle = \omega(\xi_X, \cdot)$ (the equality of elements of $T^*_x X$, both sides are just $d_x H_\xi$).

We remark that the (co)moment map is not determined uniquely.

Exercise 10.4. *Let μ, μ' be two moment maps, and X be connected. Then $\mu - \mu'$ is a constant function equal to some G-invariant element of $\mathfrak{g}^{*G}$.*

The following exercise describes some properties of the kernel and the image of $d_x\mu$.

Exercise 10.5. *Prove that $\ker d_x\mu = (\mathfrak{g}x)^\perp$ and $\operatorname{im} d_x\mu = \mathfrak{g}_x^\perp$, where in the first equality the superscript $\perp$ stands for the skew-orthogonal complement with respect to ω_x, and in the second case for the annihilator in the dual space; we write $\mathfrak{g}_x$ for the Lie algebra of stabilizer G_x. Deduce that $d_x\mu$ is surjective if and only if G_x is discrete.*

Let us consider the example of cotangent bundles. Let G act on X_0. Then this action canonically lifts to a G-action on $X = T^*X_0$ preserving α and $\omega = -d\alpha$. We claim that the assignment $H_\xi = \xi_{X_0}$ is a comoment map. Indeed, we have $\xi_X = \tilde{\xi}_{X_0}$ (the easiest way to see this is to use the C^∞-description) and, as we have already seen, $v(\xi_{X_0}) = \tilde{\xi}_{X_0}$.

Exercise 10.6. *Let $\mu : T^*X_0 \to \mathfrak{g}^*$ be the moment map. Show that $\mu^{-1}(0)$ is the union of conormal bundles to the G-orbits in X_0.*

Problem 10.7. *Let G act on a vector space V with finitely many orbits. Show that G acts on V^* with finitely many orbits and exhibit a bijection between the two sets of orbits.*

We will still need a further specialization that we have already met in Lecture 3. Take a quiver $\underline{Q} = (Q_0, \underline{Q}_1)$ and consider the double quiver $Q = (Q_0, Q_1)$, where, for each arrow $a \in \underline{Q}_1$, we add an opposite arrow a^*. Fix a dimension vector $v = (v_i)_{i \in Q_0}$ and consider the representation space $R_0 = \mathrm{Rep}(\underline{Q}, v)$. This space has a natural action of $G = \mathrm{GL}(v)(= \prod_{i \in Q_0} \mathrm{GL}(v_i))$. For each arrow a, we identify the space $\mathrm{Hom}(\mathbb{C}^{v_{h(a)}}, \mathbb{C}^{v_{t(a)}})$ with $\mathrm{Hom}(\mathbb{C}^{v_{h(a)}}, \mathbb{C}^{v_{t(a)}})^*$ by means of the trace form, $\langle A, B \rangle := \mathrm{tr}(AB)$. Also we identify the Lie algebra $\mathfrak{g} = \mathfrak{gl}(v)$ with its dual in a similar way. So $R := \mathrm{Rep}(Q, v)$ becomes identified with $R_0 \oplus R_0^* = T^*R_0$ and we can view the moment map μ as a morphism $R \to \mathfrak{g}$.

Proposition 10.1. *We have $\mu = (\mu_i)_{i \in Q_0}$, where*

$$\mu_i(x_a, x_{a^*}) = \sum_{a \in \underline{Q}_1, h(a)=i} x_a x_{a^*} - \sum_{a \in \underline{Q}_1, t(a)=i} x_{a^*} x_a.$$

This is different by the sign from what we had before.

Proof. We start with a few general properties concerning products of varieties/groups and restrictions to subgroups. Most of these properties follow from the definitions in a straightforward way.

(i) If $G_1 \times G_2$ acts on X_0, then $\mu_{G_1\times G_2}(x) = (\mu_{G_1}(x), \mu_{G_2}(x))$.
(ii) If G acts acts on $X_0 \times X_0'$, then $\mu_G(x, x') = \mu_G(x) + \mu_G(x')$ (because $\xi_{X_0\times X_0'} = (\xi_{X_0}, \xi_{X_0'})$).
(iii) Finally, if H is a subgroup of G, then $\mu_H(x) = \rho \circ \mu_G(x)$, where $\rho : \mathfrak{g}^* \to \mathfrak{h}^*$ is the restriction map.
(iv) Let V_0 be a vector space. The moment maps for $V = V_0 \oplus V_0^*$ viewed as T^*V_0 and as $T^*V_0^*$ are negative of each other (because, first, the forms are negatives of each other, and, second, we have chosen unique moment maps that are homogeneous quadratic).

The variety R_0 is the direct product of the Hom spaces. Using (ii) (and an easy part of (i) when one of the groups acts trivially) we reduce the proof to the case when $\underline{Q}_1$ has a single arrow a. Here we have two cases. First, $a : i \to j$ is not a loop and the group acting is $\mathrm{GL}(v_i) \times \mathrm{GL}(v_i)$. Second, $a : i \to i$ is a loop and the group acting is $\mathrm{GL}(v_i)$.

Let us consider the first case. By (i), we can compute μ_j and μ_i separately. Consider μ_j. We need to show that for, $A \in \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}), B \in \mathrm{Hom}(\mathbb{C}^{v_j}, \mathbb{C}^{v_i})$, we have $\mu(A, B) = AB$. We have $\xi_A = \xi A$. So $\mathrm{tr}(\mu(A, B)\xi) = \langle B, \xi A\rangle = \mathrm{tr}(B\xi A) = \mathrm{tr}(AB\xi)$ and hence $\mu(A, B) = AB$. Using (iv) we deduce that $\mu_i(A, B) = -BA$.

To get the case of a loop from the previous case we notice that the action of $\mathrm{GL}(v_i)$ on $\mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_i})$ is obtained by embedding $\mathrm{GL}(v_i)$ diagonally to $\mathrm{GL}(v_i) \times \mathrm{GL}(v_i)$. Under our identification of $\mathfrak{gl}(v_i) \cong \mathfrak{gl}(v_i)^*$, the map ρ just sends (X, Y) to $X + Y$. It remains to apply (iii). □

Problem 10.8. *Let V be a symplectic vector space with form ω and let G act on V via a homomorphism $G \to \mathrm{Sp}(V)$. Show that this action is Hamiltonian with $H_\xi(v) = \frac{1}{2}\omega(\xi v, v)$.*

The importance of moment maps in Mechanics comes from the observation that the functions H_ξ are the first integrals of any G-invariant Hamiltonian system. So all trajectories are contained in fibers of μ.

Problem 10.9. *This problem discusses symplectic forms on coadjoint orbits. Let G be an algebraic group. Pick $\alpha \in \mathfrak{g}^*$.*
(1) *Equip $T_\alpha G\alpha$ with a form ω_α by setting $\omega_\alpha(\xi_\alpha, \eta_\alpha) = \langle \alpha, [\xi, \eta]\rangle$. Prove that this is well-defined.*
(2) *Show that ω_α extends to a unique G-invariant form on $G\alpha$ (the Kirillov-Kostant form) and that this form is symplectic. Further, show that the G-action on $G\alpha$ is Hamiltonian with moment map being the inclusion.*
(3) *Let X be a homogeneous space for G equipped with a symplectic form ω such that the G-action is Hamiltonian with moment map μ. Show that the image of μ is a single orbit, say $G\alpha$, that μ is a locally trivial covering, and that ω is obtained as the pull-back of the Kirillov-Kostant form.*