LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

IVAN LOSEV

11. Calogero-Moser system and Hamiltonian reduction

11.1. **Calogero-Moser system.** The Calogero-Moser system is the system of n distinct points of the same mass, say 1, on the line (we work over $\mathbb{C}$ so we consider the complex line) with coordinates $x^1,\ldots,x^n$ that interact with pairwise potentials of the form $\frac{c}{(x^i-x^j)^2}$, where c is some nonzero constant. We can rescale and assume that $c=-1$. The total potential is $V=-\sum_{i<j}\frac{1}{(x_i-x_j)^2}$.

We want to consider the corresponding Hamiltonian system but we first need to decide what will be the symplectic variety to "accommodate" the system. Our original configuration space is $(\mathbb{C}^n)^{Reg}:=\{(x^1,\ldots,x^n)|x^i\neq x^j,\forall i\neq j\}$. So we could take the variety $X:=T^*(\mathbb{C}^n)^{Reg}$ and consider the Hamiltonian $H=\frac{1}{2}\sum_{i=1}^n y_i^2-\sum_{i<j}\frac{1}{(x_i-x_j)^2}$. However, there is a better choice. Our points are indistinguishable and so we can view $(x^1,\ldots,x^n)$ as an unordered n-tuple. So the configuration space is $(\mathbb{C}^n)^{Reg}/\mathfrak{S}_n$ and we consider its cotangent bundle $X:=T^*(\mathbb{C}^n)^{Reg}/\mathfrak{S}_n$. Also $\mathfrak{S}_n$ acts naturally on $T^*(\mathbb{C}^n)^{Reg}$, the action is induced from $(\mathbb{C}^n)^{Reg}$ and hence preserves the symplectic form. As the following exercise shows $T^*((\mathbb{C}^n)^{Reg}/\mathfrak{S}_n)=(T^*(\mathbb{C}^n)^{Reg})/\mathfrak{S}_n$. So we can view H (that is an $\mathfrak{S}_n$-invariant function) as a function on X. This is a Hamiltonian of our system. Below we will write C^{Reg} instead X. We will see that it is still not the best possible phase space for our system: we can actually embed C^{Reg} to some symplectic affine variety C (Calogero-Moser space) to "avoid collisions".

Exercise 11.1. *Let X_0 be a smooth algebraic variety equipped with a free action of a finite group Γ. Show that $T^*(X_0/\Gamma)$ is naturally identified with $(T^*X_0)/\Gamma$ (an isomorphism of symplectic varieties).*

Let us explain what kind of results regarding the Calogero-Moser (CM) system we want to get. First of all, we want to describe the trajectories, as explicitly as possible. Second, we want to produce 1st integrals subject to certain conditions. Namely, we want 1st integrals $H_1,\ldots,H_n$ with $H_2=H$ such that $\{H_i,H_j\}=0$ for all i,j and $d_xH_1,\ldots,d_xH_n$ being linearly independent for a general point $x\in X$. By some general results of Algebraic geometry (generic smoothness) the last condition is equivalent to $H_1,\ldots,H_n$ being algebraically independent. As the following exercise shows, here n is the maximal possible number for which such first integral may exist.

Exercise 11.2. *Let $f_1,\ldots,f_m$ be functions on a symplectic variety X such that $\{f_i,f_j\}=0$ for all i,j. Show that the dimension of the span of $d_xf_1,\ldots,d_xf_m$ has dimension not exceeding $\frac{1}{2}\dim X$.*

Systems admitting such collection of functions $H_1,\ldots,H_n$ (with Hamiltonian H being one of them) are called *completely integrable*. The reason is the Arnold-Liouville theorem that roughly states that (well, under some additional assumptions, and in C^∞-setting) such systems can be explicitly integrated. As we can explicitly integrate the system under consideration without using that theorem, we want to skip the details.

11.2. **Trajectories and 1st integrals of CM system.** We will follow an approach by Kazhdan, Kostant and Sternberg, [KKS], based on Hamiltonian reduction. A key observation is as follows. Pick a point $p \in T^*(\mathbb{C}^n)^{Reg}, p = (x^1, \ldots, x^n, y_1, \ldots, y_n)$ and construct two matrices, X_p, Y_p from p:

$$X_p = \operatorname{diag}(x^1, x^2, \ldots, x^n), Y_p = (y_{ij})_{i,j=1}^n, y_{ii} := y_i, y_{ij} := \frac{1}{x^i - x^j}, i \neq j.$$

The first indication that it is a reasonable thing to consider is that $H(p) = \frac{1}{2}\operatorname{tr}(Y_p^2)$. Also we notice that $[X_p, Y_p]$ has 0's on the diagonal, and 1's elsewhere (the anti-unit matrix). So $[X_p, Y_p] + E$ has only 1's and so has rank 1 and trace n. Set $O := \{A \in \operatorname{Mat}_n(\mathbb{C}) | \operatorname{tr} A = 0, \operatorname{rk}(A + E) = 1\}$, this is a single conjugacy class. Consider the adjoint $G := \operatorname{GL}(n)$-action on $R := \operatorname{Mat}_n(\mathbb{C})^2$. This is an action of the type considered in the last lecture – the action on the representation space of a double quiver. So $\mu : R \to \mathfrak{g}, \mu(X, Y) := [X, Y]$ is the moment map for this action. So we get a map $\iota : T^*(\mathbb{C}^n)^{Reg} \hookrightarrow \mu^{-1}(O), p \mapsto (X_p, Y_p)$. The image is definitely contained in the subset $\mu^{-1}(O)^{Reg}$ consisting of all $(X, Y) \in \mu^{-1}(O)$ such that X has distinct eigenvalues. We remark that $\mu^{-1}(O)^{Reg}$ is a principal open subset of $\mu^{-1}(O)$, it is the non-vanishing locus of a G-invariant polynomial.

Exercise 11.3. *Show that the action of G on $\mu^{-1}(O)^{Reg}$ is free. Also check that $\operatorname{im}\iota$ intersects any orbit and that elements $\iota(p), \iota(p')$ are G-conjugate if and only if p and p' are $\mathfrak{S}_n$-conjugate.*

Problem 11.4. *Show that the action of G on $\mu^{-1}(O)$ is free.*

This problem implies that all orbits of G in $\mu^{-1}(O)$ are closed and hence the categorical quotient $\mu^{-1}(O)//G$ coincides with the naive quotient. Also we see that $\mu^{-1}(O)^{Reg}//G$ is a principal open subset in $\mu^{-1}(O)//G$ and coincides with the naive quotient set-theoretically.

Thanks to 11.3, we can identify C^{Reg} with the naive orbit space $\mu^{-1}(O)^{Reg}/G$ (below we will see that our identification is an isomorphism of varieties). The following is our main result concerning the integration of Calogero-Moser systems.

Theorem 11.1 ([KKS])**.**
(1) *Let $p = p(0)$ be a point in $C^{Reg} = \mu^{-1}(O)^{Reg}/G$ and let $p(t)$ be its trajectory. The pairs $(X_{p(t)}, Y_{p(t)})$ and $(X_p - tY_p, Y_p)$ lie in the same G-orbit.*
(2) *The functions $H_k(p) = \operatorname{tr}(Y_p^k), k = 1, \ldots, n$ on C^{Reg} commute w.r.t. $\{\cdot, \cdot\}$ and are linearly independent thus making the CM system completely integrable.*

For this we will equip $C := \mu^{-1}(O)//G$ with a symplectic form. Then for $H \in \mathbb{C}[R]^G$ we can consider the induced function $\underline{H} \in \mathbb{C}[C]$. We will see that the trajectories for $\underline{H}$ are obtained by projecting those for H. Finally, we will show that the embedding $C^{Reg} \hookrightarrow C$ respects the symplectic forms.

The Poisson structure on $\mu^{-1}(O)/G$ is obtained by the procedure called Hamiltonian reduction. We will explain this procedure in the next two sections, first on the algebraic level and then on the geometric one.

11.3. **Hamiltonian reduction, algebraically.** Let A be a Poisson algebra and G be an algebraic group acting on A by Poisson algebra automorphisms and rationally. Since the action is rational, it differentiates to a natural Lie algebra homomorphism $\mathfrak{g} \to \operatorname{Der}(A), \xi \mapsto \xi_A$. We also require that the G-action on A is Hamiltonian, i.e., we have a G-equivariant

Lie algebra homomorphism $\xi \mapsto H_\xi : \mathfrak{g} \to A$ such that $\xi_A = \{H_\xi, \cdot\}$. Let μ^* denote the corresponding homomorphism $S(\mathfrak{g}) \to A$. Then the ideal $A\mu^*(\mathfrak{g})$ is G-stable. By the Hamiltonian reduction $A///_0G$ we mean the algebra $[A/A\mu^*(\mathfrak{g})]^G$.

The point of considering this algebra is that it comes equipped with a natural Poisson bracket. Namely, for $a+A\mu^*(\mathfrak{g}), b+A\mu^*(\mathfrak{g}) \in [A/A\mu^*(\mathfrak{g})]^G$, we set $\{a+A\mu^*(\mathfrak{g}), b+A\mu^*(\mathfrak{g})\} := \{a, b\}+A\mu^*(\mathfrak{g})$. We, first, need to check that this is well-defined, i.e., $\{A\mu^*(\mathfrak{g}), b+A\mu^*(\mathfrak{g})\} \subset A\mu^*(\mathfrak{g})$ as long as $b + A\mu^*(\mathfrak{g}) \in [A/A\mu^*(\mathfrak{g})]^G$. The latter is equivalent to $gb - b \in A\mu^*(\mathfrak{g})$ and implies $\xi_A b \in A\mu^*(\mathfrak{g})$ for all $\xi \in \mathfrak{g}$. But $\xi_A b = \{H_\xi, b\}$. So $\{A\mu^*(\mathfrak{g}), b\} \subset \{A, b\}\mu^*(\mathfrak{g}) + A\{\mu^*(\mathfrak{g}), b\}$. Both summands are contained in $A\mu^*(\mathfrak{g})$, the second because our choice of b. Also $\{A\mu^*(\mathfrak{g}), A\mu^*(\mathfrak{g})\} \subset A\mu^*(\mathfrak{g})$ because $\{\mu^*(\mathfrak{g}), \mu^*(\mathfrak{g})\} \subset \mu^*(\mathfrak{g})$. This completes the proof of the fact that the bracket on $A///_0G$ is well-defined. Since the bracket $\{a+A\mu^*(\mathfrak{g}), b+A\mu^*(\mathfrak{g})\}$ is well-defined, it is G-invariant. Indeed, $g\{a, b\} = \{ga, gb\}$ coincides with $\{a, b\}$ modulo $A\mu^*(\mathfrak{g})$. So we do get a well-defined bracket on $[A/A\mu^*(\mathfrak{g})]^G$.

This construction is not sufficient for our purposes and we will need its generalization. Let us notice that $S(\mathfrak{g})$ comes equipped with a natural Poisson bracket given by $\{\xi, \eta\} := [\xi, \eta]$ on $\mathfrak{g}$. By a Poisson ideal in a Poisson algebra B we mean an ideal I such that $\{B, I\} \subset I$.

Exercise 11.5. *Prove that a G-invariant ideal in $S(\mathfrak{g})$ is automatically Poisson. Also show that the converse is true provided G is connected.*

For example, the ideal in $\mathbb{C}[\mathfrak{g}^*] = S(\mathfrak{g})$ of any orbit in $\mathfrak{g}^*$ is Poisson. Given a Poisson ideal $I \subset S(\mathfrak{g})$ we can form the reduction $A///_IG := [A/A\mu^*(I)]^G$.

Exercise 11.6. *Equip the algebra $A///_IG$ with a natural Poisson bracket.*

11.4. **Hamiltonian reduction, geometrically.** Let us describe the geometric side of the picture. Let X be an affine symplectic variety with form ω acted on by G in a Hamiltonian way, let μ be the moment map. Let Y be a Poisson subscheme of $\mathfrak{g}^*$ (a subscheme given by a Poisson ideal I). Then the scheme-theoretic preimage $\mu^{-1}(Y)$ has algebra of functions $\mathbb{C}[X]/\mathbb{C}[X]\mu^*(I)$. So $X///_YG := \mu^{-1}(Y)//G$ is the spectrum of the Hamiltonian reduction $\mathbb{C}[X]///_IG$. Below we always assume that G is reductive.

We will need some conditions for $X///_YG$ to be a variety and to be a smooth variety.

Proposition 11.2.
1. *Let Y be a reduced locally complete intersection in $\mathfrak{g}^*$ (i.e., smooth). Suppose that every irreducible component of $\mu^{-1}(Y)$ contains a point without stabilizer. Then $\mu^{-1}(Y)$ is reduced and hence $X///_YG$ is a variety.*
2. *Suppose that Y is smooth and the action of G on $\mu^{-1}(Y)$ is free. Then $X///_YG$ is smooth.*

Proof. Recall that $d_x\mu$ is a surjection provided G_x is finite. The assumptions of (1) imply that $\operatorname{codim}_X \mu^{-1}(Y) = \operatorname{codim}_{\mathfrak{g}^*} Y$ and hence $\mu^{-1}(Y)$ is a locally complete intersection. This scheme is generically reduced, and hence, by the Serre criterium, is reduced.

Similarly, under the assumptions of (2), $\mu^{-1}(Y)$ is smooth. Now (2) follows from the general facts about free actions on smooth varieties that are stated as a lemma below. □

Lemma 11.3. *Let Z be a smooth affine algebraic variety equipped with a free action of a reductive group G. Then the quotient $Z//G$ is smooth and parameterizes G-orbits (and so coincides with the naive quotient as a set) and the morphism $\pi : Z \to Z//G$ is smooth (i.e., is a submersion at any point).*

The proof is based on an easy observation that, for any point $z \in Z$, one can choose a smooth subvariety $Z_0 \subset Z$ containing z and transversal to Gz. Then, at least in the analytic topology, there is a neighborhood Z_0^0 of $z \in Z_0$ such that the natural map $G \times Z_0^0 \to Z$ is an open embedding whose image is some G-stable neighborhood of Gz. In the algebraic category, one cannot deal with analytic topology, but can do étale one. A general statement here: the étale slice theorem of Luna can be found in [PV]. This is quite technical and we are not going to elaborate on this.

Part (2) of the previous lemma has a standard corollary that we will need in the sequel.

Corollary 11.4. *In the notation of the previous lemma,*

(1) $\mathrm{Vect}(Z//G)$ *is the quotient of* $\mathrm{Vect}(Z)^G$ *by the vertical vector fields, i.e., vector fields tangent to the fibers of* π. *This identification is provided by the map* $\pi_* : \mathrm{Vect}(Z)^G \twoheadrightarrow \mathrm{Vect}(Z//G)$.

(2) *The pull-back* π^* *identifies the space* $\Gamma(Z//G, \bigwedge^i T^*Z//G)$ *with the subspace in* $\Gamma(Z, \bigwedge^i T^*Z)^G$ *consisting of all forms that vanish on a vertical vector field.*

Let us describe the assumptions under which $X///_Y G$ is symplectic and describe the corresponding symplectic form. Assume that in (2) of the previous lemma, Y is a single orbit. Since this orbit is closed, it is of a semisimple element (under a usual identification of $\mathfrak{g}$ with $\mathfrak{g}^*$). In particular, the stabilizer G_α of $\alpha \in Y$ is a reductive subgroup. The inclusion $\mu^{-1}(\alpha) \hookrightarrow \mu^{-1}(Y)$ induced an identification $\mu^{-1}(\alpha)//G_\alpha \xrightarrow{\sim} \mu^{-1}(Y)//G$. The latter identification follows from the observation that $\mu^{-1}(Y)$ is a G-homogeneous bundle over Y with fiber $\mu^{-1}(\alpha)$ (formally, this means that the natural morphism $G \times \mu^{-1}(\alpha) \to \mu^{-1}(Y)$ is the quotient for the diagonal action of G_α; this formal description implies the identification $\mu^{-1}(\alpha)//G_\alpha \xrightarrow{\sim} \mu^{-1}(Y)//G$).

Proposition 11.5. *We retain the assumptions of the previous paragraph, in particular, assume that* Y *is a single orbit. Let* $\pi : \mu^{-1}(\alpha) \twoheadrightarrow X///_Y G$ *be the quotient morphism, and* $\iota : \mu^{-1}(\alpha) \hookrightarrow X$ *be the inclusion. There is a unique* 2*-form* $\underline{\omega}$ *on* $\mu^{-1}(\alpha)//G_\alpha$ *such that* $\pi^*(\underline{\omega}) = \iota^*(\omega)$. *This form is symplectic.*

Proof. Step 1. To establish the existence and uniqueness of $\underline{\omega}$, we use (2) of Corollary 11.4 with G_α instead of G and $Z := \mu^{-1}(\alpha)$. We need to show that $\omega_x(u, v) = 0$, when v is tangent to $\mu^{-1}(\alpha)$, while u is tangent to $G_\alpha x$. The first condition means that $d_x\mu(v) = 0$. The second condition means that $u = \xi_x$ for some $\xi \in \mathfrak{g}_\alpha$. So $\omega_x(u, v) = \omega_x(\xi_x, v) = \langle d\mu_x(v), \xi\rangle = 0$.

The form $\underline{\omega}$ is closed because $\pi^*\underline{\omega} = \iota^*\omega$ is. It remains to check that $\underline{\omega}$ is non-degenerate and that the Poisson structure induced by $\underline{\omega}$ on $\mu^{-1}(Y)//G$ agrees with the initial one.

Step 2. To prove that $\underline{\omega}$ is non-degenerate is equivalent to $\{\xi_x | \xi \in \mathfrak{g}_\alpha\} = \ker d_x\mu \cap \ker d_x\mu^\perp$. As we have seen in the previous lecture, $\ker d_x\mu = (\mathfrak{g}x)^\perp$. So what we need to prove is that if $\xi \in \mathfrak{g}$ is such that $\omega(\eta_x, \xi_x) = 0$ for all $\eta \in \mathfrak{g}$, then $\xi \in \mathfrak{g}_\alpha$. But again, $\omega(\eta_x, \xi_x) = \langle d_x\mu(\xi_x), \eta\rangle$. Since μ is a G-equivariant map, we see that $d_x\mu(\xi_x) = \xi_{\mu(x)} = \xi_\alpha = 0$. So our condition is that $\langle \xi_\alpha, \eta\rangle = 0$. Obviously, it holds for all η if and only if $\xi_\alpha = 0$, equivalently, $\xi \in \mathfrak{g}_\alpha$. □

In the next lecture we will see that the Poisson structure on $X///_Y G$ induced from ω coincides with one obtained algebraically.

References

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