LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

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15. Quotient singularities as quiver varieties

15.1. **Main theorem.** We fix $n \geqslant 1$ and a Kleinian group $\Gamma_1 \subset \mathrm{SL}_2(\mathbb{C})$. We form the wreath-product group $\Gamma_n = \mathfrak{S}_n \ltimes \Gamma_1^n$, it naturally acts on $\mathbb{C}^{2n} = (\mathbb{C}^2)^{\oplus n}$. We are going to describe the quotient singularity $\mathbb{C}^{2n}/\Gamma_n$ as a quiver variety, i.e., as a Hamiltonian reduction of the representation space of an appropriate double quiver.

Recall that from Γ_1 we can produce its McKay quiver $\underline{Q}^{MK}$ that is of affine type, its vertices numbered by $0, \ldots, r$ are in one-to-one correspondence with Γ_1-irreps, where 0 corresponds to the trivial representation. We take the quiver $\underline{Q}^{CM}$ obtained from $\underline{Q}^{MK}$ by adding an additional vertex ∞ and one arrow from ∞ to 0. Then we take the double quiver Q^{CM} of $\underline{Q}^{CM}$.

Consider the representation space $R := \mathrm{Rep}(Q^{CM}, v)$, where $v = n\delta + \epsilon_\infty$, δ being the indecomposable imaginary root (supported on the vertices $0, \ldots, r$) and ϵ_∞ is the coordinate vector at ∞. We consider the group $G := \mathrm{GL}(n\delta)$, it acts on R in a Hamiltonian way with moment map μ constructed in Lecture 10. We remark that we can consider the larger group, $\bar{G} := G \times \mathbb{C}^\times$, where $\mathbb{C}^\times$ acts on the one-dimensional space at ∞; this group still acts on R. However, the one-dimensional torus $(x\,\mathrm{id}_{\mathbb{C}^{v_i}})_{i \in \underline{Q}_0^{CM}}$ acts trivially on R. Moreover, the moment map $\bar{\mu}$ for $\bar{G}$ is recovered from μ as follows: $\bar{\mu}(r) = (\mu(r), -\sum_{i=0}^r \mathrm{tr}\,\mu(r)_i)$. So the reductions with respect to G and with respect to $\bar{G}$ are the same.

Theorem 15.1. *[Gan-Ginzburg, [GG]] The fiber $\mu^{-1}(0)$ is reduced and has codimension $\dim G$ in R.*

We first show that the codimension of $\mu^{-1}(0)$ is $\dim G$. For this we recall (Lecture 10) that $\mu^{-1}(0)$ is the union of cotangent bundles to orbits in $R_0 := \mathrm{Rep}(\underline{Q}, v)$. The codimension of any conormal bundle is $\dim R_0$. The codimension of the union of the conormal bundles is therefore $\dim R_0 + m$, where m is "the maximal number of parameters describing G-orbits in R_0". This will be defined precisely and computed below.

Then we will show that the fiber $\mu^{-1}(0)$ is reduced. For this, as we have seen in Lecture 11, it is enough to prove that each component of $\mu^{-1}(0)$ admits a free G-orbit. To achieve this, we will need an explicit description of the components. In particular, we will see that there are $n+1$ of them.

Theorem 15.2. *We have a $\mathbb{C}^\times$-equivariant isomorphism $\mu^{-1}(0)/\!/G \cong \mathbb{C}^{2n}/\Gamma_n$.*

This is a special case of [CB2, Theorem 1.1].

15.2. **Theorems on quiver representations.** First of all, let us discuss the number of parameters needed to describe representations of a quiver $\underline{Q}$ with given dimension v up to an isomorphism. Here $\underline{Q}$ is an arbitrary quiver.

We will need a stratification of $\mathrm{Rep}(\underline{Q}, v)$ by dimensions of indecomposable summands (recall that each representation has a decomposition into the direct sum of indecomposables, the multiplicities of the summands do not depend on the choice of a decomposition, this is a special case of the Krull-Schmidt theorem). Let $I(\alpha^1,\ldots,\alpha^n)$ denote the subset of $\operatorname{Rep}(\underline{Q},v)$ of all representations, whose decomposition into indecomposables contains summands of dimensions $\alpha^1,\ldots,\alpha^n$. A choice of a decomposition of the graded vector space of dimension v into the summands of dimensions $\alpha^1,\ldots,\alpha^n$ gives rise to an embedding $\prod_{i=1}^n I(\alpha^i)\hookrightarrow I(\alpha^1,\ldots,\alpha^m)$ and to a surjection $\mathrm{GL}(v)\times\prod_{i=1}^n I(\alpha^i)\twoheadrightarrow I(\alpha^1,\ldots,\alpha^m)$ that descends to a surjection

$$\mathrm{GL}(v)\times_{\prod_{i=1}^n \mathrm{GL}(\alpha^i)}\prod_{i=1}^n I(\alpha^i)\twoheadrightarrow I(\alpha^1,\ldots,\alpha^m). \tag{1}$$

Using this (and the classical algebro-geometric result that the image of a constructible subset under a morphism is constructible), one can prove by induction that $I(\alpha^1,\ldots,\alpha^n)$ is a constructible set (i.e., is a union of finitely many locally closed subvarieties) and that these subvarieties can be chosen $\mathrm{GL}(v)$-stable.

We are now ready to define $m(\alpha)$, the number of parameters needed to describe indecomposable representations of dimension α. Let Z be an irreducible algebraic variety acted on by a connected algebraic group G. For $i\geqslant 0$ consider $Z_i:=\{z\in Z|\dim Gz=i\}$, this is a locally closed subvariety. We set $m(Z):=\max_i \dim Z_i-i$. We remark that $m(Z)=0$ is equivalent to Z having only finitely many G-orbits. The definition of $m(Z)$ extends to the case when Z is a G-stable constructible subset in some G-variety. Now we set $m(\alpha)=m(I(\alpha))$. Similarly, we can define the number $m(\alpha^1,\ldots,\alpha^n):=m(I(\alpha^1,\ldots,\alpha^n))$.

Lemma 15.3. *We have* $m(\alpha^1,\ldots,\alpha^n)=\sum_{i=1}^n m(\alpha^i)$.

Proof. The inequality $m(\alpha^1,\ldots,\alpha^n)\leqslant\sum_{i=1}^n m(\alpha^i)$ is an easy consequence of (1). Let us prove the opposite inequality. We may assume that $m(\alpha^1),\ldots,m(\alpha^k)>0, m(\alpha^{k+1})=\ldots=m(\alpha^n)=0$. Let $I^0(\alpha^i), i=1,\ldots,k$ be irreducible $\mathrm{GL}(\alpha^i)$-stable locally closed subvarieties in $I(\alpha^i)$ such that $m(I^0(\alpha^i))>0$. We still have a surjection

$$\mathrm{GL}(v)\times_{\prod_{i=1}^k \mathrm{GL}(\alpha^i)\times\mathrm{GL}(\alpha^{k+1}+\ldots+\alpha^n)}\Big(\prod_{i=1}^k I^0(\alpha^i)\times I(\alpha^{k+1}+\ldots+\alpha^n)\Big)\twoheadrightarrow I(\alpha^1,\ldots,\alpha^n).$$

It is easy to see that the stabilizer in $\mathrm{GL}(v)$ of a generic element of $\prod_{i=1}^k I^0(\alpha^i)\times I(\alpha^{k+1},\ldots,\alpha^n)$ is contained in $\prod_{i=1}^k\mathrm{GL}(\alpha^i)\times\mathrm{GL}(\alpha^{k+1}+\ldots+\alpha^n)$. So the surjection above generically has finite fibers. It follows that $m(\alpha^1,\ldots,\alpha^n)=m(\alpha^1)+\ldots+m(\alpha^k)+m(\alpha^{k+1},\ldots,\alpha^n)=m(\alpha^1)+\ldots+m(\alpha^k)$. ☐

Example 15.4. Let us consider the case of a quiver with one vertex and a single loop. Here $I(\alpha^1,\ldots,\alpha^n)$ consists of matrices whose Jordan normal form has n blocks of sizes $\alpha^1,\ldots,\alpha^n$. Clearly, $m(\alpha^1,\ldots,\alpha^n)=n$.

There is a formula for $m(\alpha)$ found by Kac. Consider the quadratic function $(v,v)=\sum_{i\in\underline{Q}_0}v_i^2-\sum_{a\in\underline{Q}_1}v_{h(a)}v_{t(a)}$. A nonzero element $\alpha\in\mathbb{Z}_{\geqslant 0}^{Q_0}$ is called a *root* if $(\alpha,\alpha)\leqslant 1$. Then set $p(v)=1-(v,v)$.

Theorem 15.5. (1) *$I(\alpha)\neq\varnothing$ if and only if α is a root and $m(\alpha)=p(\alpha)$.*

(2) *there is a decomposition $I(\alpha)=\bigsqcup_{i=0}^N I^i(\alpha)$ into irreducible locally closed G-stable subvarieties such that $m(I^0(\alpha))=p(\alpha), m(I^i(\alpha))<p(\alpha)$.*

The first part is a well-known theorem of Kac. A reference for the second one can be found in the proof of [GG, Theorem 3.2.3].

Now let us describe the doubled setting. Let Q be the double of $\underline{Q}$ and $R_0 = \operatorname{Rep}(\underline{Q}, v)$, $R = \operatorname{Rep}(Q, v) = T^*R_0$. We have the moment map $\mu : R \to \mathfrak{g} := \mathfrak{gl}(v)$. From the description of $\mu^{-1}(0)$ recalled above, we see that $\dim \mu^{-1}(0) = \dim R_0 + m(R_0)$. Indeed, let $\rho : R \twoheadrightarrow R_0$ be the projection. Let $R_{0i} := \{r \in R_0 | \dim Gr = i\}$. Then $\rho^{-1}(R_{0,i}) \cap \mu^{-1}(0)$ surjects to $\rho^{-1}(R_{0,i})$ with fibers of dimensions $\dim R_0 - i$.

A one-dimensional subtorus of G acts trivially, so $\operatorname{im}\mu \subset \mathfrak{sl}(v) := \{(A_i)_{i\in Q_0} | \sum_i \operatorname{tr}(A_i) = 0\}$ and $\operatorname{codim}_R \mu^{-1}(0) \leqslant \dim \mathfrak{g} - 1$. The equality $\operatorname{codim}_R \mu^{-1}(0) = \dim \mathfrak{g} - 1$ is equivalent to

$$m(R_0) = \dim R_0 - \dim \mathfrak{g} + 1 = \sum_{a\in \underline{Q}_1} v_{t(a)}v_{h(a)} - \sum_{i\in Q_0} v_i^2 + 1 = p(v).$$

On the other hand, from the discussion above, we see that $m(R_0) = \max \sum_{i=1}^n p(\alpha^i)$, where the max is taken over all decompositions $v = \alpha^1 + \ldots + \alpha^n$ into the sum of roots.

Theorem 15.6. *The following conditions are equivalent.*

(1) $\operatorname{codim}_R \mu^{-1}(0) = \dim \mathfrak{g} - 1$ *(this includes the claim that fiber is non-empty).*
(2) $p(v) \geqslant \sum_{i=1}^n p(\alpha^i)$ *for all decompositions* $v = \sum_{i=1}^n \alpha^i$ *into the sum of roots* α^i.

Both $\mathbb{C}[R], \mathbb{C}[\mathfrak{sl}(v)]$ are positively graded and μ is homogeneous, we now can apply a graded analog of [E, Theorem 18.16] to see that μ is flat. Being flat, μ is open, and, being in addition $\mathbb{C}^\times$-equivariant, it is surjective.

Now let us explain why we need part 2 of Theorem 15.5. Assume the equivalent conditions of Theorem 15.6 hold. It follows from Theorem 15.5 and the proof of Lemma 15.3 that one can decompose $I(\alpha^1, \ldots, \alpha^n)$ into the union of locally closed irreducible G-stable subvarieties $\bigsqcup_{j\geqslant 0} I^j(\alpha^1, \ldots, \alpha^n)$ such that $m(I^0(\alpha^1, \ldots, \alpha^n)) = \sum_{i=1}^n p(\alpha^i) > m(I^j(\alpha^1, \ldots, \alpha^n))$ for $j > 0$. Consider the subvariety $\rho^{-1}(I^j(\alpha^1, \ldots, \alpha^n)) \cap \mu^{-1}(0)$. Being a vector bundle over an irreducible variety, the intersection is irreducible. Its dimension is $\leqslant \dim R_0 + \sum_{i=1}^n p(\alpha^i)$ with equality achieved only if $j = 0$. Each irreducible component of $\mu^{-1}(0)$ contains exactly one dense $\rho^{-1}(I^j(\alpha^1, \ldots, \alpha^n)) \cap \mu^{-1}(0)$. We see that the irreducible components of $\mu^{-1}(0)$ are in one-to-one correspondence with decompositions $v = \sum_{i=1}^n \alpha^i$ such that $p(v) = \sum_{i=1}^n p(\alpha^i)$.

Below it will be sometimes convenient to deal with preprojective algebras. Recall that the preprojective algebra for Q is the quotient of the path algebra $\mathbb{C}Q$ by the relations

$$\sum_{a\in \underline{Q}_1, h(a)=i} aa^* - \sum_{a\in \underline{Q}_1, t(a)=i} a^*a = 0,$$

one for each $i \in Q_0$. Of course, $\operatorname{Rep}(\Pi^0(Q), v) = \mu^{-1}(0)$.

15.3. **Codimension.** Now we return to the case when $\underline{Q} = \underline{Q^{CM}}$. Consider the decomposition $n\delta + \epsilon_\infty = \sum_{i=0}^m \alpha^i$ into the sum of roots, where $\alpha_\infty^0 = 1$ and $\alpha_\infty^i = 0$ for $i > 0$. So α^i is a root in the corresponding affine root system.

Let p^{MK} denote the p-function for the McKay quiver. We have $p(\alpha^i) = p^{MK}(\alpha^i)$. The latter is zero when α is a real root, and 1 when α^i is a multiple of δ. Further, we have $p(n\delta + \epsilon_\infty) = p^{MK}(n\delta) - 1 + n = 1 - 1 + n = n$.

Now we prove $p(n\delta + \epsilon_\infty) \geqslant \sum_{i=0}^m p(\alpha^i)$ and that the equality holds in exactly one of the following situations: $\alpha^0 = k\delta + \epsilon_\infty, \alpha^1 = \ldots = \alpha^{n-k} = \delta$ for some $k = 0, \ldots, n$.

We have $p(\alpha^0)=\alpha^0_0+p^{MK}(\alpha^0-\epsilon_\infty)-1$. We have $p^{MK}(\alpha^0-\epsilon_\infty)\leqslant 1$ with equality only if $\alpha^0-\epsilon_\infty=k\delta$. So either $p(\alpha^0)<\alpha^0_0$ or $p(\alpha^0)=\alpha^0_0$ for $\alpha^0=k\delta+\epsilon$. We also have $p(\alpha^i)\leqslant\alpha^i_0$ with equality only if $\alpha^i=\delta$. Since $\sum_{i=0}^m\alpha^i_0=n$, we are done.

This already proves the claim about codimension. Also this proves that the total number of irreducible components is $n+1$.

15.4. **Points without stabilizer.** We will describe the $n+1$ components of $\mu^{-1}(0)\subset\operatorname{Rep}(Q,n\delta+\epsilon_\infty)$ explicitly and in each we produce a point with a trivial stabilizer. But first we need to determine simple representations in $\mu^{-1}(0)$ for some other dimension vectors.

Lemma 15.7. *Let v be a dimension vector for Q^{MK}.*

1. *If $v<\delta$ (i.e., $v\neq\delta$ and all coordinates of $\delta-v$ are non-negative), then the only semi-simple representation in* $\operatorname{Rep}(\Pi^0(Q^{MK}),v)$ *is* 0.
2. *If $v=\delta$, then* $\operatorname{Rep}(\Pi^0(Q^{MK}),v)$ *is irreducible and a generic representation is simple.*

Proof. It is enough to prove the claim for the simple representations. The dimension of all components of $\operatorname{Rep}(\Pi^0(Q^{MK}),v)$ is $\sum_{a\in Q_1^{MK}}v_{t(a)}v_{h(a)}$. If there is a non-zero simple representation, then, due to $\mathbb{C}^\times$-equivariance, there is a one-parameter family of such, each with G-orbit of dimension $\sum_{i=0}^r v_i^2-1$. So we see that $0\leqslant\dim\mu^{-1}(0)-\dim G=-(v,v)<0$, contradiction.

Let us now consider the case of $v=\delta$. Then there is only one component of $\operatorname{Rep}(\Pi^0(Q^{MK}),\delta)$ of dimension $\sum_a\delta_{t(a)}\delta_{h(a)}+1$. This is proved by analogy with the previous section. Since $\operatorname{Rep}(\Pi^0(Q^{MK}),\delta)/\!/\operatorname{GL}(\delta)\cong\mathbb{C}^2/\Gamma_1$, we see that there are infinitely many isomorphism classes of semi-simple representations. On the other hand, by (1), any reducible nonzero semisimple representation is 0. So any representation lying in the complement of the zero fiber of $\operatorname{Rep}(\Pi^0(Q^{MK}),\delta)\to\operatorname{Rep}(\Pi^0(Q^{MK}),\delta)/\!/\operatorname{GL}(\delta)$ is simple. ☐

Take pairwise distinct simple representations $x_1,\ldots,x_n$ of $\operatorname{Rep}(\Pi^0(Q^{MK}),\delta)$. Pick a decomposition of $\bigoplus_{i=0}^r\mathbb{C}^{n\delta_i}$ into $(\bigoplus\mathbb{C}^{\delta_i})^{\oplus n}$. Then $x:=\bigoplus_{i=1}^n x_i$ is in $\operatorname{Rep}(\Pi^0(Q^{MK}),n\delta)$. The stabilizer of x in G is isomorphic to $(\mathbb{C}^\times)^n\hookrightarrow\operatorname{GL}(\delta)^{\times n}\hookrightarrow\operatorname{GL}(n\delta)$. It acts on $\mathbb{C}^n$ (the space of maps corresponding to the arrow from ∞ to 0) faithfully by diagonal matrices, let $e_1,\ldots,e_n$ be an eigenbasis. Consider the locally closed subvariety $\mathcal{M}_k:=\{(x_1,\ldots,x_n,i,j)\}$, where $x_1,\ldots,x_n$ are as above, $i\in\mathbb{C}^n$ a vector that is the span of $e_1,\ldots,e_k$ with nonzero coefficients, $j\in\mathbb{C}^{n*},j(e_1)=\ldots=j(e_k)=0,j(e_{k+1}),\ldots,j(e_n)\neq0$. In particular, we see that $ij=0$ and so $\mathcal{M}_k\subset\mu^{-1}(0)$. The stabilizer of (i,j) in $\mathbb{C}^{\times n}$ is trivial and so the stabilizer of any point in $\mathcal{M}_k$ is trivial. We claim that $\overline{G\mathcal{M}_k}$ are different irreducible components of $\mu^{-1}(0)$. It is easy to see that $G\mathcal{M}_k\cap G\mathcal{M}_{k'}=\varnothing$ for $k\neq k'$ (just consider the (i,j) components). Clearly, $\mathcal{M}_k$ is stable under $\operatorname{GL}(\delta)^{\times n}$ and the action of this group is free. The dimension of the quotient is the number of parameters for the x_ℓ's and this number is $2n$. The map

$$\operatorname{GL}(n\delta)\times_{\operatorname{GL}(\delta)^{\times n}}\mathcal{M}_k\to\mu^{-1}(0),(g,m)\mapsto gm$$

has finite fibers (that are orbits for a natural action of $\mathfrak{S}_k\times\mathfrak{S}_{n-k}$). So $\dim\overline{G\mathcal{M}_k}=\dim G+2n=\dim\mu^{-1}(0)$. Our claim is proved and this finishes the proof of Theorem 15.1.

15.5. **Sketch of proof of Theorem 15.2.** In fact, one can construct a morphism $\mathbb{C}^{2n}/\Gamma_n=(\mathbb{C}^2/\Gamma_1)^n/\mathfrak{S}_n\to\mu^{-1}(0)/\!/G$ and then prove that this is an isomorphism.

Recall that $\mathbb{C}^2/\Gamma_1=\mu_1^{-1}(0)/\!/\operatorname{GL}(\delta)$, where $\mu_1:\operatorname{Rep}(Q^{MK},\delta)\to\mathfrak{gl}(\delta)$ is the moment map. We have a map $[\mu_1^{-1}(0)/\!/\operatorname{GL}(\delta)]^n\to\mu^{-1}(0)/\!/G$ induced by $(x_1,\ldots,x_n)\in\mu_1^{-1}(0)^n\mapsto$

$(x_1 \oplus \ldots \oplus x_n, 0, 0)$. Since permuting the summands does not change the G-orbit, this morphism descends to $\psi : [\mu_1^{-1}(0)/\!/ \operatorname{GL}(\delta)]^n/\mathfrak{S}_n \to \mu^{-1}(0)/\!/G$.

We claim that this morphism is bijective. This amounts to showing that every semisimple representation of in $\operatorname{Rep}(Q, n\delta+\epsilon_\infty)$ decomposes into the sum $x_1\oplus\ldots\oplus x_n\oplus(0,0)$, where $x_k \in \mu_1^{-1}(0)$ (and then $x_1, \ldots, x_n$ are defined uniquely up to isomorphisms and a permutation). This is a consequence of the following theorem of Crawley-Boevey describing the possible dimension vectors of simple representations in $\mu^{-1}(0)$ together with our computations in Section 3.

Theorem 15.8. *Let Q be a double quiver of $\underline{Q}$, v be its dimension vector. Then the following statements are equivalent.*

(1) *There is a simple representation in* $\operatorname{Rep}(\Pi^0(Q), v)$.
(2) $p(v) > \sum_{i=1}^m p(\alpha^i)$ *for any proper decomposition of v into the sum of roots.*

By the construction ψ is $\mathbb{C}^\times$-equivariant. The $\mathbb{C}^\times$-actions on both varieties contract everything to 0. Since the preimage of 0 under ψ is a single point, we deduce that ψ is finite, this is a geometric version of the graded Nakayama lemma.

The variety $\mathbb{C}^{2n}/\Gamma_n$ is normal. There is a general result of Crawley-Boevey, [CB3], saying that $\mu^{-1}(0)/\!/ \operatorname{GL}(v)$ is normal for any double quiver Q and any dimension vector v. So in our case the variety $\mu^{-1}(0)/\!/G$ is normal, and this completes the proof.

References

[CB1] W. Crawley-Boevey, *Geometry of the moment map for representations of quivers,* Comp. Math. **126** (2001), 257–293.

[CB2] W. Crawley-Boevey, *Decomposition of Marsden-Weinstein reductions for representations of quivers*, Comp. Math. **130** (2002), 225–239.

[CB3] W. Crawley-Boevey, *Normality of Marsden-Weinstein reductions for representations of quivers*, Math. Ann. 325 (2003), 55-79.

[E] D.Eisenbud, *Commutative algebra with a view towards algebraic geometry.* GTM 150, Springer Verlag, 1995.

[GG] W.L. Gan, V. Ginzburg, *Almost commuting variety, $\mathcal{D}$-modules and Cherednik algebras.* IMRP, 2006, doi: 10.1155/IMRP/2006/26439. arXiv:math/0409262.