LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

IVAN LOSEV

Correction to Section 15.4

Unfortunately, the argument in Section 15.4 of the lecture that shows that all $n+1$ components of $\mu^{-1}(0)$ contain a point with a trivial stabilizer is incorrect. The reason is that only $\mathcal{M}_0, \mathcal{M}_n$ are subvarieties in $\mu^{-1}(0)$, while the other $n-1$ varieties $\mathcal{M}_i$ even do not intersect $\mu^{-1}(0)$. A correct argument is below.

Lemma 0.1. *A generic representation in* $\operatorname{Rep}(\underline{Q}^{MK}, \delta)$ *is indecomposable and its stabilizer in* $\operatorname{GL}(\delta)$ *is* $\mathbb{C}^\times$.

Proof. All subsets $I(\alpha^1, \ldots, \alpha^n) \subset \operatorname{Rep}(\underline{Q}^{MK}, \delta)$ with $n > 1$ contain finitely many orbits. Let us decompose $I(\delta)$ into locally closed irreducible G-stable subvarieties, $I(\delta) = I^0(\delta) \sqcup I^1(\delta) \sqcup \ldots \sqcup I^k(\delta)$ with $m(I^0(\delta)) = 1$ and $m(I^\ell(\delta)) = 0$ for $\delta > 0$. This means that we may assume that all $I^j(\delta)$ are single G-orbits. The dimension of every orbit does not exceed $\dim \operatorname{GL}(\delta) - 1$. We have $\dim \underline{R} = \dim \operatorname{GL}(\delta)$. So we see that $I^0(\delta)$ is dense in $\underline{R}$. Also a dimension count shows that a generic orbit in $I^0(\delta)$ has to have dimension $\dim \operatorname{GL}(\delta) - 1$. The stabilizer of every representation is connected (it is an open subset in the space of all endomorphisms of the representation). So we see that the stabilizer of a generic representation in $I^0(\delta)$ is forced to coincides with $\mathbb{C}^\times$, the kernel of the $\operatorname{GL}(\delta)$-action. $\square$

Recall from Section 15.3, that the components of $\mu^{-1}(0)$ are the closures of the conormal bundles to some locally closed subsets $I^k \subset I(k\delta + \epsilon_\infty, \delta, \ldots, \delta) \subset \operatorname{Rep}(\underline{Q}^{CM}, n\delta + \epsilon_\infty)$ with $m(I^k) = n$. We will now present such subsets. Namely, let $\underline{x}_1, \ldots, \underline{x}_n$ be pairwise distinct indecomposable elements from $\operatorname{Rep}(\underline{Q}^{MK}, \delta)$ with stabilizer $\mathbb{C}^\times$. Such representations exist thanks to the previous lemma. Choose a decomposition $\bigoplus_{i \in \underline{Q}_0^{MK}} \mathbb{C}^{n\delta_i} = (\bigoplus_{i \in \underline{Q}_0^{MK}} \mathbb{C}^{\delta_i})^{\oplus n}$. Then we can view $\underline{x} := \bigoplus_{j=1}^n \underline{x}_j$ as an element of $\operatorname{Rep}(\underline{Q}^{MK}, n\delta)$. Also we have the induced decomposition of the space $\mathbb{C}^n$ sitting at the vertex 0 into the direct sum of one-dimensional subspaces. Let $e_1, \ldots, e_n$ be a basis compatible with this decomposition. To get an element of $\operatorname{Rep}(\underline{Q}^{CM}, n\delta + \epsilon_\infty)$ from an element of $\operatorname{Rep}(\underline{Q}^{MK}, n\delta)$ we need to add an element of $\mathbb{C}^n$. Set $I^k := \{\underline{x}, i := \sum_{\ell=1}^k i_\ell e_\ell | i_1 \ldots i_k \neq 0\}$, where $\underline{x}$ is as above. The stabilizer of $\underline{x}$ in G is $\mathbb{C}^{\times n} \hookrightarrow \operatorname{GL}(\delta)^{\times n} \hookrightarrow G = \operatorname{GL}(n\delta)$. So the stabilizer of $(\underline{x}, i) \in I^k$ is $\mathbb{C}^{\times(n-k)}$, the last $n-k$ copies of $\mathbb{C}^\times$ in $\mathbb{C}^{\times n}$. We need to show that the stabilizer in $\mathbb{C}^{\times(n-k)}$ of a generic point of the fiber in $(\underline{x}, i)$ of the conormal bundle to $G(\underline{x}, i)$ is trivial.

The space $\mathfrak{g}(\underline{x}, i) = T_x G(\underline{x}, i)$ admits an epimorphism onto $\mathfrak{g}\underline{x}$ with kernel $\mathfrak{g}_{\underline{x}} i$. Clearly, $\mathfrak{g}_{\underline{x}} i = \operatorname{Span}(e_1, \ldots, e_k)$. So the conormal space to the orbit $G(\underline{x}, i)$ naturally surjects onto $(\mathbb{C}^n / \operatorname{Span}(e_1, \ldots, e_k))^*$. The action of $(\mathbb{C}^\times)^{n-k}$ on the latter space is faithful and so it is faithful on the whole conormal space implying, in particular, that the stabilizer of a generic point is trivial.