LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

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16. Symplectic resolutions of $\mu^{-1}(0)//G$ and their deformations

16.1. **GIT quotients.** We will need to produce a resolution of singularities for $\mathbb{C}^{2n}/\Gamma_n = \mu^{-1}(0)//G$. Such resolutions may be constructed as GIT quotients for the action of G on $\mu^{-1}(0)$. In this section we are going to recall a few basics about such quotients.

Let X be an affine algebraic variety acted on by a reductive group G. The categorical quotient $X//G$ parameterizes closed orbits for the G-action. It is insufficient for many applications (e.g., in constructing moduli spaces in Algebraic geometry) as one needs to parameterize other types of orbits. One thing one can try to do is to choose some G-stable affine open subsets $X_1, \ldots, X_n$ of X, take their categorical quotients $X_i//G$ and then glue the quotients together (the quotients $X_i//G$ and $X_j//G$ are glued along $(X_i \cap X_j)//G$). However, the result generally will be a nasty space (not an algebraic variety). There is however one setting, where we get a nice variety that often appears to be even better than the usual categorical quotient.

Namely, fix a character θ of G. We say that $x \in X$ is *θ-semi-stable* if there is $f \in \mathbb{C}[X]^{G,n\theta} := \{f \in \mathbb{C}[X] | g.f = \theta(g)^n f\}$ for some $n > 0$ and $f(x) \neq 0$. A not semi-stable point is called *unstable*. A semistable point with finite stabilizer is usually called *stable*. The locus of θ-semistable points is denoted by $X^{\theta-ss}$. By definition, it can be covered by principal open subsets X_f, where f is as above. We can take the categorical quotients $X_f//G$ and glue them together in a natural way. The result of gluing is the Proj of the graded algebra $\bigoplus_{n=0}^{+\infty} \mathbb{C}[X]^{G,n\theta}$ (the product is restricted from $\mathbb{C}[X]$, the degree n component, by definition, is $\mathbb{C}[X]^{G,n\theta}$) denoted by $X//^\theta G$. By the construction, there is a natural projective morphism $X//^\theta G \to X//G$. We remark that for $\theta = 0$, all points are semi-stable, and $X//^\theta G = X//G$. We also remark that, for $m > 0$, the sets of semistable points for θ and for $m\theta$ coincide, and $X//^\theta G = X//^{m\theta} G$.

For example, we can consider the action of $\mathbb{C}^\times$ on $\mathbb{C}^2$ by $t.(x_1, x_2) = (tx_1, tx_2)$. If $\theta(t) = t^{-1}$, then $\mathbb{C}[X]^{G,n\theta} = \mathbb{C}[x_1, x_2]^n$ and $X//^\theta G = \mathbb{P}^1$. If $\theta(t) = t$, then $\mathbb{C}[X]^{G,n\theta} = \{0\}$, and $X^{\theta-ss} = \varnothing$.

Exercise 16.1. *Consider the G-action on $X \times \mathbb{C}$ given by $g(x, z) = (gx, \theta(g)z)$. Show that $x \in X^{ss}$ iff $\overline{G(x, 1)}$ doesn't intersect $X \times \{0\}$.*

16.2. **Case of quivers.** As an application of the previous exercise we will compute semi-stable point in $\operatorname{Rep}(Q, v)$, where Q is a quiver, under the action of $\operatorname{GL}(v)$. Any character θ is given by a Q_0-tuple $(\theta_i)_{i \in Q_0}, \theta((g_i)_{i \in Q_0}) = \prod_{i \in Q_0} \det(g_i)^{\theta_i}$. We remark that if $\theta \cdot v := \sum_{i \in Q_0} \theta_i v_i \neq 0$, then $\mathbb{C}[\operatorname{Rep}(Q, v)]^{\operatorname{GL}(v),\theta} = 0$ (the semi-invariant space can only be nonzero if the character vanishes on the kernel of the action). Below we only consider θ with this property.

Proposition 16.1. *A representation of Q with dimension v is θ-semistable iff it does not contain a subrepresentation with dimension vector v' subject to $\theta \cdot v' > 0$.*

Proof. Set $U := \bigoplus_{i\in Q_0}\mathbb{C}^{v_i}$, and let x be a representation of Q in this space. Let U' be a subrepresentation of dimension v' with $\theta\cdot v'>0$. Choose a complement U''. Consider the one-parametric subgroup $\gamma:\mathbb{C}^\times\to \mathrm{GL}(v)$ defined by $\gamma(t)=(t1_{U'},1_{U''})$. Clearly $\lim_{t\to 0}\gamma(t)x$ exists. Also, since $\theta\cdot v'>0$, we see that $\lim_{t\to 0}\theta(\gamma(t))=0$. It follows that $\lim_{t\to 0}(\gamma(t)x,\theta(\gamma(t)))$ exists and lies in $\mathrm{Rep}(Q,v)\times\{0\}$. By the previous exercise, x is not semi-stable.

Conversely, assume x is not semi-stable. According to the Hilbert-Mumford criterium (see Lecture 3), there is a one-parameter subgroup $\gamma:\mathbb{C}^\times\to\mathrm{GL}(v)$ such that $\lim_{t\to 0}(\gamma(t)x,\theta(\gamma(t)))$ exists and lies in $\mathrm{Rep}(Q,v)\times\{0\}$. Let U^j be the eigen-subspace for $\gamma(t)$ in U with eigencharacter $t\mapsto t^j$, let v^j be the dimension of U^j. The equality $\lim_{t\to 0}\theta(\gamma(t))=0$ is equivalent to $\sum_j j\theta\cdot v^j>0$. Equivalently, $\sum_k \theta\cdot(v^k+v^{k+1}+\ldots)>0$. On the other hand, since $\lim_{t\to 0}\gamma(t)x$ exists, $\bigoplus_{j\geqslant k}U^j$ is a subrepresentation. We are done. $\square$

16.3. **Nakajima quiver varieties.** Fix a quiver $\underline{Q}$, and $v,w\in\mathbb{Z}_{\geqslant 0}^{\underline{Q}_0}$. Also fix vector spaces $V_i,W_i, i\in\underline{Q}_0,\dim V_i=v_i,\dim W_i=w_i$. Then consider the space

$$R_0=\mathrm{Rep}(\underline{Q},v,w):=\bigoplus_{a\in\underline{Q}_1}\mathrm{Hom}(V_{t(a)},V_{h(a)})\oplus\bigoplus_{i\in\underline{Q}_0}\mathrm{Hom}(W_i,V_i),$$

its double $R=T^*R_0$ and the action of $G:=\mathrm{GL}(v)$ on these spaces. Before we considered the case when $\underline{Q}$ is the McKay quiver, $v=n\delta$, and $w=\epsilon_0$. As before, $\mathrm{Rep}(\underline{Q},v,w)=\mathrm{Rep}(\underline{Q}^w,v+\epsilon_\infty)$, where $\underline{Q}^w$ is the quiver obtained from $\underline{Q}$ by adding the new vertex ∞ together with w_i arrows from ∞ to i. We extend a character θ of G to that $\mathrm{GL}(v+\epsilon_\infty)=G\times\mathbb{C}^\times$ by $\bar\theta(g,t)=\theta(g)t^{-v\cdot\theta}$.

Consider the moment map $\mu:R\to\mathfrak{g}$ and form the Hamiltonian reduction $\mathcal{M}^\theta(Q,v,w):=\mu^{-1}(0)/\!/^\theta G$. The action of $\mathbb{C}^\times$ on R by dilations descends to $\mathcal{M}^\theta(Q,v,w)$.

Nakajima proved (this is not a very deep result; it follows from an alternative point of view on the reductions $\mu^{-1}(0)/\!/^\theta G$ – via so called *hyper-Kähler reduction*) that the action of G on $\mu^{-1}(0)^{ss}$ is free iff θ is *generic* in the sense that $\theta\cdot v'\neq 0$ for all $v'\leqslant v$. In particular, θ with $\theta_i=-1$ for all i is generic. So, for generic θ, $\mathcal{M}^\theta(Q,v,w)$ is a smooth symplectic variety with form, say, ω. This form is compatible with the $\mathbb{C}^\times$-action: it gets rescaled, $t.\omega=t^2\omega$.

Exercise 16.2. *Prove that for $\theta=(-1)_{i\in Q_0}$, the subset R^{ss} consists of all elements*

$$(x_a,x_{a^*},y_i,z_i)_{a\in\underline{Q}_1,i\in\underline{Q}_0}$$

(here $x_a\in\mathrm{Hom}(V_{t(a)},V_{h(a)}),x_{a^*}\in\mathrm{Hom}(V_{h(a)},V_{t(a)}),y_i\in\mathrm{Hom}(W_i,V_i),z_i\in\mathrm{Hom}(V_i,W_i)$) such that there are no proper subspaces $V_i'\subset V_i$ that are stable under all x_a,x_{a^*} and such that $\operatorname{im}y_i\subset V_i'$. Deduce that the action of $\mathrm{GL}(v)$ on $\mathrm{Rep}(Q,v,w)^{\theta-ss}$ is free.

By the definition of $\mathcal{M}^\theta(Q,v,w)$ we have a projective morphism $\rho:\mathcal{M}^\theta(Q,v,w)\to\mathcal{M}^0(Q,v,w)$, where the target scheme is affine. This morphism is $\mathbb{C}^\times$-equivariant and Poisson. Often, this morphism happens to be a resolution of singularities. For example, this is so when $\mu^{-1}(0)$ contains a free closed orbit and θ is generic. In the case of interest for us (affine $\underline{Q},v=n\delta,w=\epsilon_0$) this does not hold. However, ρ is still a resolution of singularities. This is true for an arbitrary generic θ but we will only prove the statement for the choice of θ above. We will need to use a general fact about the varieties $\mathcal{M}^\theta(Q,v,w)$: they are connected. This follows from the claim that $\mathcal{M}^\theta(Q,v,w)$ is diffeomorphic to $\mu^{-1}(\lambda)/\!/G$ for a generic $\lambda\in\mathfrak{g}^{*G}$ (due to Nakajima, this again follows from the hyper-Kähler reduction approach) and a theorem of Crawley-Boevey that $\mu^{-1}(\lambda)$ is connected (see a problem below proving this in our case).

Proposition 16.2. *Suppose $\underline{Q}$ is an affine quiver, $v = n\delta, w = \epsilon_0, \theta = (-1)_{i\in\underline{Q}_0}$. Then ρ is a resolution of singularities.*

Proof. Since the G-action on R^{ss} is free, we see that

$$\dim \mathcal{M}^\theta(Q, v, w) = \dim \mu^{-1}(0)^{ss} - \dim G = \dim R - 2\dim G = 2n = \dim \mathbb{C}^{2n}/\Gamma_n =$$
$$= \dim \mathcal{M}^0(Q, v, w).$$

The only thing that we need to prove is

(*) for a generic point $p \in \mathcal{M}^0(Q, v, w)$, the fiber $\rho^{-1}(p)$ is a single point.

We claim that R^{ss} intersects not more than one component of $\mu^{-1}(0)$. Indeed, any free orbit in $\mu^{-1}(0)$ lies in a single component (because μ is a submersion at any point of a free orbit). The variety $\mathcal{M}^\theta(Q, v, w)$ is connected and hence so is $R^{ss} \cap \mu^{-1}(0)$. Since $R^{ss} \cap \mu^{-1}(0)$ is connected, we see that it lies in a single component of $\mu^{-1}(0)$ and is dense there for dimension reasons (if non-empty). Further, we claim that the component is of the form $\overline{\{(x, i, 0)\}}$ with $i \in \mathbb{C}^n$ and x being a generic semisimple representation in $\mathrm{Rep}(Q^{MK}, n\delta)$. Indeed, it is easy to show that all points $\{(x, i, 0)\}$ with x, i generic satisfy the conditions of Exercise 16.2. The claim (*) boils down to showing that, for fixed generic $x \in \mathrm{Rep}(\Pi^0(Q^{MK}), \delta)$, all points $\{(x, i, 0)\} \in R^{ss}$ lie in the same G-orbit. Equivalently, we need to show that all i such that $(x, i, 0) \in R^{ss}$ lie in the same G_x-orbit. As we have seen in Lecture 15, $G_x = (\mathbb{C}^\times)^n$ acting on $\mathbb{C}^n$ in the usual way. We know that G acts on R^{ss} freely and so G_x acts on the set of i with $(x, i, 0) \in R^{ss}$ freely. But there is only one free orbit of $(\mathbb{C}^\times)^n$ in $\mathbb{C}^n$. We are done. $\square$

Problem 16.3. *Prove that a generic fiber $\mu^{-1}(\lambda), \lambda \in \mathfrak{g}^{*G}$, is smooth and connected. You may use the following strategy:*

1. *Show that all G-orbits in $\mu^{-1}(\lambda)$ are free. Deduce that $\mu^{-1}(\lambda)$ is smooth.*
2. *Show that $\mu^{-1}(\mathbb{C}\lambda)$ is normal.*
3. *The affine version of Zariski's main theorem says that the morphism $\mu^{-1}(\mathbb{C}\lambda) \to \mathbb{C}\lambda$ decomposes into a composition of a morphism $\mu^{-1}(\mathbb{C}\lambda) \to Y$ with connected general fibers and a finite morphism $Y \to \mathbb{C}\lambda$. Use this to prove that $\mu^{-1}(\lambda)$ is connected.*

Example 16.3. Consider the example, when $\underline{Q}$ is a Jordan quiver. Here $\mu^{-1}(0)^{ss}$ consists of all triples (X, Y, i) with $X, Y \in \mathrm{Mat}_n(\mathbb{C}), i \in \mathbb{C}^n$ such that $[X, Y] = 0$ and i is a cyclic vector for (X, Y): $\mathbb{C}[X, Y]i = \mathbb{C}^n$. To such a triple we can assign an ideal of codimension n in $\mathbb{C}[x, y]$ consisting of all polynomials $f \in \mathbb{C}[x, y]$ such that $f(X, Y) = 0$. The scheme $\mu^{-1}(0)/\!/^\theta G$ is therefore the Hilbert scheme $\mathrm{Hilb}_n(\mathbb{C}^2)$ parameterizing such ideals in $\mathbb{C}[x, y]$.

16.4. **Deformations of GIT Hamiltonian reductions.** Now suppose that V is a symplectic vector space and a connected reductive group G acts on V via a homomorphism $G \to \mathrm{Sp}(V)$. Let $\mu : V \to \mathfrak{g}^*$ be the moment map and $\Phi : \mathfrak{g} \to W_\hbar(V)$ be the quantum comoment map, where $W_\hbar(V)$ is the Weyl algebra. Let θ be a character of G such that G acts freely on $\mu^{-1}(0)^{ss}$. Recall the notation $\mathfrak{z} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$.

Recall that under some additional assumption the universal quantum Hamiltonian reduction $W_\hbar(V)/\!/\!/G := [W_\hbar(V)/W_\hbar(V)\Phi([\mathfrak{g}, \mathfrak{g}])]^G$ is a graded flat deformation of $V/\!/\!/_0 G$ (more precisely, of the corresponding algebra of functions) over the algebra $\mathbb{C}[\mathfrak{z}^*, \hbar]$.

We are going to produce a deformation $\mathcal{D}$ of $X := V/\!/\!/^\theta G$. This variety is not affine, it cannot be defined using a single algebra of functions. Rather, it is defined by its sheaf of regular functions $\mathcal{O}_X$. Let us recall some details on this sheaf, as this will serve a motivation for the definition of $\mathcal{D}$.

The variety X is covered by the open subsets $V_f///G$ with $f \in \mathbb{C}[V]^{G,n\theta}$, where V_f is the principal open subset associated to f. By definition, the space $H^0(V_f///G, \mathcal{O}_X)$ of sections of $\mathcal{O}_X$ is $\mathbb{C}[V_f]///_0G$. For $f_1 \in \mathbb{C}[V]^{G,n_1\theta}, f_2 \in \mathbb{C}[V]^{G,n_2\theta}$ we have $(V_{f_1}///_0G) \cap (V_{f_2}///_0G) = V_{f_1f_2}///_0G$ and the restriction map $H^0(V_{f_1}///G, \mathcal{O}_X) \to H^0(V_{f_1f_2}///G, \mathcal{O}_X)$ is induced from the inclusion $\mathbb{C}[V_{f_1}] = \mathbb{C}[V][f_1^{-1}] \hookrightarrow \mathbb{C}[V_{f_1f_2} = \mathbb{C}[V][f_1^{-1}f_2^{-1}]$. We also remark that X can be covered by the subsets $V_f///_0G$, where f is homogeneous (with respect to the action of $\mathbb{C}^\times$ by dilations). This is because all homogeneous component of $f \in \mathbb{C}[V]^{G,n\theta}$ are again in $\mathbb{C}[V]^{G,n\theta}$.

The deformation $\mathcal{D}$ will be a $\mathbb{C}^\times$-equivariant sheaf of algebras $\mathcal{D}$ over $\mathbb{C}[[\mathfrak{z}^*, \hbar]]$, flat, complete and separated in the $(\mathfrak{z}^*, \hbar)$-adic topology. The precise meaning of the word "deformation" in this setting is $\mathcal{D}/(\mathfrak{z}, \hbar) = \mathcal{O}_X$. The deformation is produced using a suitable (="sheaf") version of the quantum Hamiltonian reduction.

Namely, for $f \in \mathbb{C}[V]^{G,n\theta}$ we can consider the localization $W_\hbar(V)[f^{-1}]$ that is $\hbar$-adically complete and separated and is a flat deformation of $\mathbb{C}[V][f^{-1}]$, see Exercise 14.4. The group G still acts on $W_\hbar(V)[f^{-1}]$ and Φ is a quantum comomoment map. Consider the left ideal $W_\hbar(V)[f^{-1}]\Phi([\mathfrak{g},\mathfrak{g}]) = [\mathrm{W}_\hbar(V)\Phi([\mathfrak{g},\mathfrak{g}])]\,[f^{-1}]$.

Problem 16.4. *Show that the algebra $W_\hbar(V)[f^{-1}]$ is Noetherian. Deduce from here that any left ideal is closed in the $\hbar$-adic topology.*

Also we claim that $\mathcal{M}_\hbar := \mathrm{W}_\hbar(V)/W_\hbar(V)[f^{-1}]\Phi([\mathfrak{g},\mathfrak{g}])$ is $\hbar$-flat. The proof is a modification of that of Proposition 14.1. The argument there does not apply directly in our setting: a result on regular sequences used there works for positively graded or local rings. However, one can reduce to the case of local rings as follows. Let K be the kernel of the $\hbar$-action on $\mathcal{M}_\hbar[f^{-1}]$. If K is nonzero, then so is its stalk at some point $x \in \mu^{-1}(0)^{ss}$. On the other hand, we can localize at x, and we get $(\mathcal{M}_\hbar)_x = W_\hbar(V)_x/W_\hbar(V)_x\Phi([\mathfrak{g},\mathfrak{g}])$. Now the argument of the proposition works and we see that $(\mathcal{M}_\hbar)_x$ is flat over $\mathbb{C}[[\hbar]]$. So $K_x = 0$, a contradiction.

So we get a a flat deformation $\mathrm{W}_\hbar(V)[f^{-1}]///G = \mathcal{M}_\hbar^G$ of

$$\mathbb{C}[V][f^{-1}]///G := (\mathbb{C}[V][f^{-1}]/\mathbb{C}[V][f^{-1}]\mu^*([\mathfrak{g},\mathfrak{g}]))^G.$$

The algebra $\mathrm{W}_\hbar(V)[f^{-1}]///G$ is complete and separated in the $\hbar$-adic topology. This follows from Problem 16.4. For two elements $f_1 \in \mathbb{C}[V]^{n_1\theta}, f_2 \in \mathbb{C}[V]^{n_2\theta}$ we have a natural homomorphism $W_\hbar(V)[f_1^{-1}]///G \to W_\hbar(V)[f_1^{-1}f_2^{-1}]///G$ induced by $W_\hbar(V)[f_1^{-1}] \hookrightarrow \mathrm{W}_\hbar(V)[f_1^{-1}f_2^{-1}]$.

Exercise 16.5. *Reducing modulo $\hbar^k$ for all k, show that the data of $W_\hbar(V)[f^{-1}]///G$ constitutes a sheaf on $V///^\theta G := \mu^{-1}(\mathfrak{z}^*)//^\theta G$ with respect to the covering $V_f///G$ of $V///^\theta G$. Recall that this means the following: if we have a covering of $V_f///G$ by $V_{f_i}///G, i = 1, \ldots, n$ and sections $a_1, \ldots, a_n$ of $W_\hbar(V)[f_i^{-1}]///G$ that agree on intersections, then they glue together to a unique element $W_\hbar(V)[f^{-1}]///G$.*

So we can glue the algebras $W_\hbar(V)[f^{-1}]///G$ to a sheaf on $X///^\theta G$ denoted by $W_\hbar(V)///^\theta G$. This sheaf is $\mathbb{C}^\times$-equivariant: if $f \in \mathbb{C}[V]^{G,n\theta}$ is homogeneous, the algebra $W_\hbar(V)[f^{-1}]///G$ has a natural $\mathbb{C}^\times$-action. Since these actions are compatible with the restriction homomorphisms, they give rise to a $\mathbb{C}^\times$-action on the sheaf $W_\hbar(V)///^\theta G$. By the construction, $W_\hbar(V)///^\theta G/(\hbar) = \mathcal{O}_{V///^\theta G}$.

The sheaf $W_\hbar(V)///^\theta G$ is basically a sheaf we need, but we want to make it $(\mathfrak{z}, \hbar)$-adically complete and leaving on $X = \mu^{-1}(0)//^\theta G$, rather than on $V///^\theta G$. This is a pretty formal procedure, we pull-back $W_\hbar(V)///^\theta G$ to $\mu^{-1}(0)//^\theta G \subset V///^\theta G$ sheaf-theoretically and then complete the resulted sheaf with respect to the $(\mathfrak{z}, \hbar)$-adic topology, to get the required sheaf

$\mathcal{D}$. In more pedestrian terms, the sections of $\mathcal{D}$ on $V_f///_0 G$ is the $(\mathfrak{z}, \hbar)$-adic completion of $W_\hbar(V)[f^{-1}]///G$. Since the natural morphism $V///^\theta G \to \mathfrak{z}^*$ is flat near 0, we see that $\mathcal{D}$ is flat over $\mathbb{C}[[\mathfrak{z}^*, \hbar]]$.