

# LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

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## 17. PROCESI BUNDLES AND THEIR DEFORMATIONS

**17.1. Recap.** Let  $V$  denote a symplectic vector space and let  $G$  be a reductive group acting on  $V$  via a homomorphism  $G \rightarrow \mathrm{Sp}(V)$ . This gives rise to the moment map  $\mu : V \rightarrow \mathfrak{g}^*$ , a  $G$ -action on the homogenized Weyl algebra  $W_h(V)$ , and a quantum comoment map  $\Phi : \mathfrak{g} \rightarrow W_h(V)$  that equals to  $\mu^*$  modulo  $\hbar$ . Let  $\mathfrak{z}$  denote the quotient  $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ . Pick a character  $\theta$  of  $G$ .

We make the following assumptions:

- (i) Every irreducible component of  $\mu^{-1}(0)$  contains a free  $G$ -orbit.
- (ii) The action of  $G$  on  $\mu^{-1}(0)^{\theta-ss}$  is free.

Under assumption (i), the quantum Hamiltonian reduction

$$W_h(V)///G = [W_h(V)/W_h(V)\Phi([\mathfrak{g}, \mathfrak{g}])]^G$$

is a graded deformation of  $\mathbb{C}[V///_0 G]$  over  $\mathbb{C}[\mathfrak{z}^*, \hbar]$ . Under assumption (ii), the variety  $X := \mu^{-1}(0)^{\theta-ss}/G$  is smooth and symplectic. Furthermore, it comes with a deformation  $\mathcal{D}$  of  $\mathcal{O}_X$ , where  $\mathcal{D}$  is a sheaf of  $\mathbb{C}[\mathfrak{z}^*, \hbar]$ -algebras. This deformation is flat (over  $\mathbb{C}[\mathfrak{z}^*, \hbar]$ ) and complete and separated in the  $(\mathfrak{z}, \hbar)$ -adic topology. Below we will abbreviate this as “FCS deformation over  $\mathbb{C}[\mathfrak{z}^*, \hbar]$ ”. Moreover,  $\mathcal{D}$  is  $\mathbb{C}^\times$ -equivariant.

Let us recall how the deformation  $\mathcal{D}$  is constructed. It is enough to specify the sections and the restriction homomorphisms on some base of topology on  $X$ . For a base, we can take  $X_f := V_f///_0 G$ , where  $f \in \mathbb{C}[V]^{G, n\theta}$  and  $V_f$  stands for the principal open subset of  $V$  defined by  $f$ . By definition, the algebra of section  $\mathcal{D}(X_f)$  of  $\mathcal{D}$  on  $X_f$  is the  $(\mathfrak{z}, \hbar)$ -adic completion of  $W_h(V)[f^{-1}]/G$ , where  $W_h(V)[f^{-1}]$  is the completed localization of  $W_h(V)$  with respect to  $f$ , see Exercise 14.4. For  $f_1 \in \mathbb{C}[V]^{G, n_1\theta}, \mathbb{C}[V]^{G, n_2\theta}$ , we have a natural homomorphism of algebras  $W_h(V)[f_1^{-1}] \rightarrow W_h(V)[f_1^{-1}f_2^{-1}]$  that gives rise to a homomorphism  $W_h(V)[f_1^{-1}]/G \rightarrow W_h(V)[f_1^{-1}f_2^{-1}]/G$  that, in its turn, produces a homomorphism  $\mathcal{D}(X_{f_1}) \rightarrow \mathcal{D}(X_{f_1f_2})$ . This is a restriction homomorphism we need.

To establish the  $\mathbb{C}^\times$ -equivariant structure on  $\mathcal{D}$ , it is sufficient to construct a  $\mathbb{C}^\times$ -action on  $\mathcal{D}(X_f)$  with homogeneous  $f \in \mathbb{C}[V]^{G, n\theta}$  (the corresponding subsets  $X_f$  also cover  $X$ ) and make sure that the restriction homomorphisms are  $\mathbb{C}^\times$ -equivariant. We take the  $\mathbb{C}^\times$ -action on  $\mathcal{D}(X_f)$  induced from the  $\mathbb{C}^\times$ -action on  $W_h(V)[f^{-1}]$ , such action is compatible with restriction homomorphisms.

Finally, let us remark that we have a natural  $\mathbb{C}^\times$ -equivariant and  $\mathbb{C}[\mathfrak{z}^*, \hbar]$ -equivariant homomorphism  $W_h(V)///G \rightarrow \mathcal{D}(X)$ . Indeed, the homomorphisms  $W_h(V) \rightarrow W_h(V)[f^{-1}]$  give rise to homomorphisms  $W_h(V)///G \rightarrow \mathcal{D}(X_f)$  that agree on intersections and hence glue to a homomorphism  $W_h(V)///G \rightarrow \mathcal{D}(X)$ .

**17.2. Deformations of sheaves.** Let  $X$  be an algebraic variety (or scheme),  $\mathcal{F}_0$  be a coherent sheaf on  $X$  and let  $\mathcal{D}$  be a FCS deformation of  $X$  over the formal power series ring  $\mathbb{C}[[x_1, \dots, x_n]]$ . We are interested in the following two questions. First, is there a deformation

$\mathcal{F}$  of  $\mathcal{F}_0$  to a flat (and then, in fact, automatically complete and separated) right  $\mathcal{D}$ -module and if so, is such deformation unique? Second, how are the global sections of  $\mathcal{F}$  and  $\mathcal{F}_0$  are related, in particular, whether the former is a deformation of the latter? To ensure affirmative answers to our questions, we will need to impose some cohomology vanishing assumptions.

Here is an answer to the first question.

**Lemma 17.1.** *Let  $X$  be an algebraic variety,  $\mathcal{F}_0$  be a coherent sheaf on  $X$  with  $H^i(X, \mathcal{F}_0) = 0$  for  $i > 0$ . Let  $\mathcal{F}$  be a sheaf of  $\mathcal{D}$ -modules, flat over  $\mathbb{C}[[x_1, \dots, x_n]]$ , complete and separated in the  $(x_1, \dots, x_n)$ -adic topology and such that  $\mathcal{F}/(x_1, \dots, x_n) = \mathcal{F}_0$ . Then  $H^p(X, \mathcal{F}) = 0$  for  $p > 0$  and  $\mathcal{F}(X)$  is a FCS deformation of  $\mathcal{F}_0(X)$ .*

*Proof.* Before we prove the lemma in the general case, we need to understand the case when  $X$  is affine. Here the cohomology vanishing for  $\mathcal{F}_0$  is automatic. Given a finitely generated  $\mathcal{D}(X)$ -module  $F$  we can sheafify it to a sheaf  $\mathcal{F}$  over  $X$  (so that  $\mathcal{F}(X_f) = \mathcal{D}(X)[f^{-1}] \otimes_{\mathcal{D}(X)} F$  for any  $f \in \mathbb{C}[X]$ ). Similarly to the case of usual sheaves, the sheafification functor is an equivalence of categories (the target category is that of finite generated sheaves of modules over  $\mathcal{D}$ ), a quasi-inverse equivalence is provided by taking global sections. And, as in the usual algebro-geometric story, the Čech cohomology of  $\mathcal{F}$  are derived functors to the functor of taking global sections. Since the latter is exact, the higher Čech cohomology vanish.

Let us return to the general case. Thanks to the previous paragraph, we can compute the Čech cohomology using open affine coverings. Set  $\mathcal{F}_i = \mathcal{F}/(x_{i+1}, \dots, x_n)\mathcal{F}$ . We will prove by induction on  $i$  that  $H^p(X, \mathcal{F}_{i+1}) = 0$  for  $p > 0$  and that  $\mathcal{F}_{i+1}(X)$  is a flat deformation of  $\mathcal{F}_i(X)$ . Our base is  $i = -1$ , where the second claim is vacuous.

Choose a covering of  $X$  by open affine subsets,  $X = \bigcup_j X_j$ . The intersection of two affine open subsets is affine. So, for  $X_{j_1, \dots, j_m} = \bigcap_{k=1}^m X_{j_k}$ , we have  $\mathcal{F}_{i+1}(X_{j_1, \dots, j_m})/x_{i+1}\mathcal{F}_{i+1}(X_{j_1, \dots, j_m}) = \mathcal{F}_i(X_{j_1, \dots, j_m})$ .

Take a cocycle  $a_\bullet \in \bigoplus_{j_1, \dots, j_m} \mathcal{F}_{i+1}(X_{j_1, \dots, j_m})$ . It is a cocycle also modulo  $x_{i+1}$ . So, since  $H^m(X, \mathcal{F}_i) = 0$ , there are  $b_\bullet^0 \in \bigoplus_{j_1, \dots, j_{m-1}} \mathcal{F}_{i+1}(X_{j_1, \dots, j_{m-1}})$ ,  $a_\bullet^1 \in \mathcal{F}_{i+1}(X_{j_1, \dots, j_m})$  such that  $a_\bullet = db_\bullet^0 + x_{i+1}a_\bullet^1$ . We have  $0 = da_\bullet = d^2b_\bullet + d(x_{i+1}a_\bullet^1) = x_{i+1}da_\bullet^1$ . Since  $\mathcal{F}_{i+1}$  is flat over  $\mathbb{C}[[x_{i+1}]]$ , we see that  $da_\bullet^1 = 0$ . We can repeat the same procedure with  $a_\bullet^1$ , getting elements  $b_\bullet^1, a_\bullet^2$ , etc. So  $a_\bullet = d(b^0 + x_{i+1}b^1 + \dots)$  (the element in the right hand side is well-defined because  $\mathcal{F}_{i+1}$  is complete and separated in the  $x_{i+1}$ -adic topology). This proves  $H^m(X, \mathcal{F}_{i+1}) = 0$ .

Being a  $\mathbb{C}[[x_{i+1}]]$ -submodule in  $\bigoplus_j \mathcal{F}_{i+1}(X_j)$ , the  $\mathbb{C}[[x_{i+1}]]$ -module  $\mathcal{F}_{i+1}(X)$  is flat and separated in the  $x_{i+1}$ -adic topology. So it remains to prove that  $\mathcal{F}_{i+1}(X)/x_{i+1}\mathcal{F}_{i+1}(X) = \mathcal{F}_i(X)$  (by induction, this will also show that  $\mathcal{F}_{i+1}(X)$  is FCS deformation of  $\mathcal{F}_0(X)$  over  $\mathbb{C}[[x_1, \dots, x_{i+1}]]$ ). Since  $\mathcal{F}_{i+1}$  is a deformation of  $\mathcal{F}_i$  (as a sheaf), we only need to prove that the natural map  $\mathcal{F}_{i+1}(X) \rightarrow \mathcal{F}_i(X)$  is surjective.

Pick a global section  $s \in \mathcal{F}_i(X)$ . The restriction  $s_i$  of  $s$  to  $X_j$  lifts to  $\bar{s}_j \in \mathcal{F}_{i+1}(X_j)$ . Then  $d\bar{s}_\bullet$  is a cocycle that vanishes modulo  $x_{i+1}$ . So there is a cocycle  $(c_\bullet) \in \bigoplus_j \mathcal{F}_{i+1}(X_j)$  such that  $d\bar{s}_\bullet = x_{i+1}c_\bullet$ . Since  $H^1(X, \mathcal{F}_{i+1}) = 0$ , we see that  $c_\bullet$  is a coboundary,  $c_\bullet = dc'_\bullet$ . Therefore  $d(\bar{s}_\bullet - x_{i+1}c'_\bullet) = 0$  meaning that  $(\bar{s} - x_{i+1}c')$  is a global section of  $\mathcal{F}_{i+1}$  that projects to  $s$ .  $\square$

Now let us give some answer to the first question: on existence and uniqueness of a deformation.

**Lemma 17.2.** *Let  $\mathcal{F}_0$  be a locally free coherent sheaf on  $X$  such that  $H^i(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F}_0)) = 0$ . Let  $\mathcal{D}$  be a FCS deformation of  $\mathcal{O}_X$  over  $\mathbb{C}[[x_1, \dots, x_n]]$ . Then the following is true.*

- (1) *There is a unique  $\mathbb{C}[[x_1, \dots, x_n]]$ -flat deformation  $\mathcal{F}$  of  $\mathcal{F}_0$  to a right  $\mathcal{D}$ -module.*
- (2) *We have  $H^i(X, \mathcal{E}nd_{\mathcal{D}^{opp}}(\mathcal{F})) = 0$  and  $\text{End}_{\mathcal{D}^{opp}}(\mathcal{F})$  is a FCS deformation of  $\text{End}_{\mathcal{O}_X}(\mathcal{F}_0)$ .*

Here  $\mathcal{E}nd$  stands for the sheaf of endomorphisms and  $\text{End}$  for the algebra of endomorphisms, the global sections of the sheaf of endomorphisms (and also the space of endomorphisms in the corresponding category).

We are not going to prove Lemma 17.2, we will just make some remarks.

First of all, the condition that  $\mathcal{F}_0$  is locally free implies that  $H^i(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F}_0)) = \text{Ext}_{\mathcal{O}_X}^i(\mathcal{F}_0, \mathcal{F}_0)$ . Now part (a) (in the case when  $n = 1$ , the general case is obtained by induction) can be deduced from the following problem.

**Problem 17.1.** *Let  $X$  be an algebraic variety,  $\mathcal{F}_0$  be a coherent sheaf on  $X$  and  $\mathcal{D}$  be a FCS (=flat, complete and separated) deformation of  $\mathcal{O}_X$  over  $\mathbb{C}[[\hbar]]$ .*

- (1) *Show that the category of finitely generated modules (i.e., sheaves) over  $\mathcal{D}/(\hbar^n)$  has enough injective objects. How are the injectives for different  $n$  related?*
- (2) *Show that if  $\text{Ext}^2(\mathcal{F}_0, \mathcal{F}_0) = 0$ , then there exists a flat deformation of  $\mathcal{F}_0$  to a right module  $\mathcal{F}_n$  over  $\mathcal{D}/(\hbar^{n+1})$ . Moreover, show that these deformations may be chosen in a compatible way and so give rise to a FCS deformation  $\mathcal{F}$  of  $\mathcal{F}_0$  to a right module over  $\mathcal{D}$ .*
- (3) *Finally, show that if  $\text{Ext}^1(\mathcal{F}_0, \mathcal{F}_0) = 0$ , then all the deformations above are unique.*

Also the condition that  $\mathcal{F}_0$  is locally free shows that  $\mathcal{E}nd_{\mathcal{D}^{opp}}(\mathcal{F})$  is a deformation of  $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{F}_0)$  and so part (b) follows from part (a).

Finally, let us mention that a straightforward analog of Lemma 17.2 holds when a  $\mathbb{C}^\times$ -action is present ( $\mathbb{C}^\times$  acts on  $X$  and both  $\mathcal{F}_0$  and  $\mathcal{D}$  are  $\mathbb{C}^\times$ -equivariant; the resulting sheaf  $\mathcal{F}$  is also then equivariant). To see it one either adjusts the previous problem to the  $\mathbb{C}^\times$ -equivariant setting or just uses the uniqueness of  $\mathcal{F}$ . In more detail, for  $t \in \mathbb{C}^\times$  we can define the push-forward  $t_*\mathcal{F}$  with respect to the action of  $t$ . This sheaf is also a deformation so that  $t_*\mathcal{F} \cong \mathcal{F}$ . This yet does not imply that  $\mathcal{F}$  is equivariant, one needs to show that isomorphisms  $t_*\mathcal{F} \cong \mathcal{F}$  are compatible with the product on  $\mathbb{C}^\times$ . The latter can be deduced from Hilbert's theorem 90.

**17.3. Global sections.** We want to understand a relationship between  $\mathcal{D}$  and the algebra  $W_\hbar(V)///G$ . We can take the algebra  $\mathcal{D}(X)$  of global sections of  $\mathcal{D}$ . This algebra carries the action of  $\mathbb{C}^\times$  and is complete in the  $(\mathfrak{z}, \hbar)$ -adic topology. We recall that there is a natural morphism  $W_\hbar(V)///G \rightarrow \mathcal{D}(X)$ . This morphism is  $\mathbb{C}^\times$ -equivariant and  $\mathbb{C}[\mathfrak{z}^*, \hbar]$ -linear.

**Theorem 17.3.** *Suppose that  $V///_0G$  is a normal scheme and that the natural morphism  $\rho : X \rightarrow V///_0G$  is a resolution of singularities. Then  $W_\hbar(V)///G$  is dense in  $\mathcal{D}(X)$  and coincides with the subspace of  $\mathbb{C}^\times$ -finite elements in  $\mathcal{D}(X)$ .*

*Proof. Step 1.* We claim that  $\mathbb{C}[X] = \mathbb{C}[V///_0G]$  (the isomorphism is  $\rho^*$ ) and  $H^i(X, \mathcal{O}_X) = 0$  for  $i > 0$ . The former follows from the claim that  $\mathbb{C}[X]$  is a finite birational extension of  $\mathbb{C}[V///_0G]$ . The cohomology vanishing follows from the observation that, being symplectic,  $X$  has trivial canonical bundle by applying the Grauert-Riemenschneider vanishing theorem.

*Step 2.* Thanks to Step 1, we can apply Lemma 17.1 to  $\mathcal{F}_0 = \mathcal{O}_X$  and  $\mathcal{F} = \mathcal{D}$ . We get that  $\mathcal{D}(X)$  is a FCS deformation of  $\mathbb{C}[X]$ .

*Step 3.* The homomorphism  $W_h(V)///G \rightarrow \mathcal{D}(X)$  extends to the  $(\mathfrak{z}, \hbar)$ -adic completion  $(W_h(V)///G)^\wedge$  of  $W_h(V)///G$ . The resulting homomorphism is the identity modulo  $(\mathfrak{z}, \hbar)$  and both algebras in consideration are complete and flat. The claim that  $(W_h(V)///G)^\wedge \xrightarrow{\sim} \mathcal{D}(X)$  now follows from the next exercise.

**Exercise 17.2.** *Let  $V_1, V_2$  be  $\mathbb{C}[[\mathfrak{z}^*, \hbar]]$ -modules that are flat, complete and separated. Let  $\iota : V_1 \rightarrow V_2$  be a  $\mathbb{C}[[\mathfrak{z}^*, \hbar]]$ -module homomorphism that is an isomorphism modulo  $(\mathfrak{z}, \hbar)$ . Show that  $\iota$  is an isomorphism.*

*Step 4.* It remains to show that  $W_h(V)///G$  coincides with the  $\mathbb{C}^\times$ -finite part of  $(W_h(V)///G)^\wedge$ . This is an easy consequence of the observation that  $W_h(V)///G$  is positively graded, see the following exercise.

**Exercise 17.3.** *Let  $A_0$  be a  $\mathbb{Z}_{\geq 0}$ -graded vector space and  $A$  be its FCS deformation over  $\mathbb{C}[[x_1, \dots, x_n]]$ . Equip  $A$  with a  $\mathbb{C}^\times$ -action such that  $t.(x_i a) = t^2 x_i t.a$  and the projection  $A \twoheadrightarrow A_0$  is  $\mathbb{C}^\times$ -equivariant (where the action of  $\mathbb{C}^\times$  on the  $i$ th component of  $A_0$  is by  $t \mapsto t^i$ ). Show that the  $\mathbb{C}^\times$ -finite part of  $A$  is a graded deformation of  $A_0$  over  $\mathbb{C}[x_1, \dots, x_n]$ .*

□

The previous theorem can be informally stated as: the deformation  $W_h(V)///G$  of  $V///_0 G$  can be lifted to its resolution of singularities  $X$ .

**17.4. Procesi bundles.** We return to the setting of Lecture 15:  $V = T^* \text{Rep}(\underline{Q}^{MK}, n\delta, \epsilon_0)$ ,  $G = \text{GL}(n\delta)$ . All assumptions made above (the assumptions (i) and (ii) in Section 1 and the assumptions of Theorem 17.3) hold.

Let  $X$  be a  $\mathbb{C}^\times$ -equivariant symplectic resolution of  $V///_0 G = \mathbb{C}^{2n}/\Gamma_n$  (where the  $\mathbb{C}^\times$ -action and the form  $\omega$  on  $X$  are related via  $t.\omega = t^2\omega, t \in \mathbb{C}^\times$ ). For example, we can take  $X = V///_0^\theta G$  for generic  $\theta$ , conjecturally there are no other possibilities.

By a *Procesi bundle* on  $X$  we mean a  $\mathbb{C}^\times$ -equivariant vector bundle  $\mathcal{P}$  satisfying the following conditions.

- (P1)  $\text{End}_{\mathcal{O}_X}(\mathcal{P})$  is an algebra over  $\mathbb{C}[X] = \mathbb{C}[\mathbb{C}^{2n}]^{\Gamma_n}$  that comes equipped with a natural  $\mathbb{C}^\times$ -action. We require the  $\mathbb{C}[\mathbb{C}^{2n}]^{\Gamma_n}$ -algebras  $\text{End}_{\mathcal{O}_X}(\mathcal{P})$  and  $\mathbb{C}[\mathbb{C}^{2n}] \# \Gamma_n$  to be  $\mathbb{C}^\times$ -equivariantly isomorphic.
- (P2)  $H^i(X, \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{P})) = 0$  for  $i > 0$ .

There is yet one more condition. To state it we remark that an endomorphism algebra of a vector bundle acts on every fiber. In particular, we have a fiberwise action of  $\Gamma_n$  on  $\mathcal{P}$ .

**Exercise 17.4.** *Show that any fiber of a Procesi bundle is isomorphic to  $\mathbb{C}\Gamma_n$  as a  $\Gamma_n$ -module.*

In particular, the bundle  $\mathcal{P}^{\Gamma_n}$  of  $\Gamma_n$ -invariants has rank 1. Clearly, (P1) and (P2) are preserved by multiplication by a line bundle. We impose a normalization condition:

- (P3)  $\mathcal{P}^{\Gamma_n} = \mathcal{O}_X$ .

The only result we need about Procesi bundles is the following.

**Theorem 17.4** ([BK]). *A Procesi bundle exists.*

We notice that a Procesi bundle is never unique. This follows, for example, from the following problem.

**Problem 17.5.** *Show that the dual of a Procesi bundle is again a Procesi bundle.*

A more subtle statement is that a Procesi bundle cannot be isomorphic to its dual (as a  $\mathbb{C}^\times$ -equivariant bundle).

Let us discuss origins of Procesi bundles and motivations to consider them. This discussion will not be used below.

There is a case when it is easy to construct Procesi bundles:  $n = 1$ . Here a Procesi bundle can be obtained as a tautological bundle on  $X = V//^\theta G$ . Namely, consider the  $G$ -equivariant vector bundle  $N$  on  $V$  that is trivial with fiber  $\bigoplus_{i \in Q_0^{MK}} (\mathbb{C}^{\delta_i})^{\oplus \delta_i}$  as a usual vector bundle and is equipped with a natural fiberwise action of  $G$ . We restrict  $N$  to  $\mu^{-1}(0)^{ss}$  and then push it forward to  $X$  getting a vector bundle of rank  $|\Gamma_1|$ . The conditions (P1),(P2) were checked in [KV]. Their motivation was to interpret the McKay correspondence as an equivalence of derived categories. Namely, in the general case of  $\Gamma_n$ , a Procesi bundle  $\mathcal{P}$  gives rise to an equivalence  $\mathrm{RHom}(\mathcal{P}, \bullet) : D^b(\mathrm{Coh}(X)) \rightarrow D^b(\mathbb{C}[\mathbb{C}^{2n}] \# \Gamma_n - \mathrm{mod})$  of triangulated categories.

The first construction of  $\mathcal{P}$  in the case when  $\Gamma_n = \mathfrak{S}_n$  is due to Haiman, [H1], see also review [H2]. Motivation of Haiman to consider Procesi bundles had an entirely different origins: algebraic combinatorics, more precisely the study of Macdonald polynomials. These are certain symmetric polynomials depending on the independent variables  $q$  and  $t$ , parameterized by partitions  $\lambda$  and defined “axiomatically” – as only polynomials subject to certain conditions. They are linear combinations of Schur polynomials with coefficients in  $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ . The Macdonald positivity conjecture says that the coefficients are actually in  $\mathbb{Z}_{\geq 0}[q^{\pm 1}, t^{\pm 1}]$ . This conjecture was first proved by Haiman using the Procesi bundle on  $\mathrm{Hilb}_n(\mathbb{C}^2)$ .

On  $\mathrm{Hilb}_n(\mathbb{C}^2)$  we have an action of  $(\mathbb{C}^\times)^2$  (that can be viewed as induced from the action on  $\mathbb{C}^2$ ). The dilation actions considered above is obtained by embedding  $\mathbb{C}^\times$  to  $(\mathbb{C}^\times)^2$  diagonally. There are finitely many fixed points for the  $(\mathbb{C}^\times)^2$ -action on  $\mathrm{Hilb}_n(\mathbb{C}^2)$ , they are precisely the monomial ideals and so are parameterized by partitions  $\lambda$  of  $n$ , let  $x_\lambda$  be the point corresponding to  $\lambda$ . Haiman’s Procesi bundle happens to be  $(\mathbb{C}^\times)^2$ -equivariant. So the fiber  $\mathcal{P}_\lambda$  of  $\mathcal{P}$  at  $x_\lambda$  is a bigraded  $\mathfrak{S}_n$ -module. Its Frobenius character is a linear combination of Schur polynomials with coefficients in  $\mathbb{Z}_{\geq 0}[q^{\pm 1}, t^{\pm 1}]$ . Haiman proved that these are Macdonald polynomials thus proving the positivity conjecture.

Haiman’s proof does not use any very advanced machinery but is very complicated. Several other proofs of the positivity conjecture including combinatorial ones are available. Some of them are based on alternative constructions of Procesi bundles. There is a proof due to Gordon, [Go], based on a construction of a Procesi bundle due to Ginzburg, [Gi]. Also there is a proof by Bezrukavnikov and Finkelberg, [BF], based on the construction of a Procesi bundle from [BK]. This proof generalizes to the case when  $\Gamma_n = \mathfrak{S}_n \ltimes (\mathbb{Z}/\ell\mathbb{Z})$  and instead of the usual Macdonald polynomials one considers so called *wreath* Macdonald polynomials.

**17.5. Isomorphism of quantum Hamiltonian reduction and spherical SRA.** Recall that our goal was to establish an isomorphism of  $W_h(V)//G$  and  $eHe$ . Recall that  $W_h(V)//G$  is a graded algebra over  $\mathbb{C}[\mathfrak{z}^*, \hbar]$ , where  $\mathfrak{z}$  and  $\hbar$  have degree 2. Similarly,  $eHe$  is a graded algebra over  $P$ , where  $P$  is a vector space with basis  $\hbar, c_0, c_1, \dots, c_r$ , all these elements are in degree 2. We have  $W_h(V)//G/(\mathfrak{z}, \hbar) = eHe/(P) = \mathbb{C}[\mathbb{C}^{2n}]^{\Gamma_n}$ . So we want a graded algebra isomorphism  $eHe \rightarrow W_h(V)//G$  that maps  $P$  to  $\mathfrak{z} \oplus \mathbb{C}\hbar$  and induces the identity on  $\mathbb{C}[\mathbb{C}^{2n}]^{\Gamma_n}$ .

We will prove a weaker statement. Namely, following [L], we will see that there is a map  $\nu : P \rightarrow \mathfrak{z} \oplus \mathbb{C}\hbar$  such that  $\mathbb{C}[\mathfrak{z}, \hbar] \otimes_{S(P)} eHe \cong W_h(V)//G$ . A crucial tool here is a deformation of the Procesi bundle.

By Lemma 17.2(1), there is a unique deformation  $\widehat{\mathcal{P}}$  of  $\mathcal{P}$  to a right  $\mathcal{D}$ -module. By part (2) of the same lemma,  $\text{End}_{\mathcal{D}^{opp}}(\widehat{\mathcal{P}})$  is a FCS deformation of  $\mathbb{C}[\mathbb{C}^{2n}]\#\Gamma_n$  over  $\mathbb{C}[[\mathfrak{z}^*, \hbar]]$ . Exercise 17.3 shows that the subalgebra  $H' \subset \text{End}_{\mathcal{D}^{opp}}(\widehat{\mathcal{P}})$  of  $\mathbb{C}^\times$  finite elements is a graded deformation of  $\mathbb{C}[\mathbb{C}^{2n}]\#\Gamma_n$ . But  $H$  is a universal graded deformation of  $\mathbb{C}[\mathbb{C}^{2n}]\#\Gamma_n$  (at least in the case when  $\Gamma_n \neq \mathfrak{S}_n$ , in the case of the symmetric group one needs to modify the argument slightly). So we get a linear map  $\nu_{\mathcal{P}} : P \rightarrow \mathfrak{z}^* \oplus \mathbb{C}\hbar$  such that  $\mathbb{C}[\mathfrak{z}^*, \hbar] \otimes_{S(P)} H \cong H'$ .

We remark that we still have an inclusion  $\Gamma_n \subset H'$ . We claim that  $eH'e$  is naturally identified with  $W_{\hbar}(V)///G$ . To see this we remark that we have a decomposition  $\widehat{\mathcal{P}} = e\widehat{\mathcal{P}} \oplus (1-e)\widehat{\mathcal{P}}$  and that  $e\widehat{\mathcal{P}}$  is a deformation of  $e\mathcal{P} = \mathcal{O}_X$ . From the uniqueness of a deformation, we see that  $e\widehat{\mathcal{P}} = \mathcal{D}$ . Next, on one hand,  $\text{End}_{\mathcal{D}^{opp}}(e\widehat{\mathcal{P}}) = e\text{End}_{\mathcal{D}^{opp}}(\widehat{\mathcal{P}})e$  and on the other hand,  $\text{End}_{\mathcal{D}^{opp}}(e\widehat{\mathcal{P}}) = \mathcal{D}(X)$ . Taking finite elements and using Theorem 17.3, we arrive at  $eH'e = W_{\hbar}(V)///G$ . So  $\mathbb{C}[\mathfrak{z}, \hbar] \otimes_{S(P)} eHe \cong W_{\hbar}(V)///G$ .

One can get an explicit formula for  $\nu$  that also shows that it is an isomorphism. We need some more notation to write it down.

Let  $N_0, \dots, N_r$  be the irreducible  $\Gamma_1$ -modules with  $N_0$  being the trivial module. Let  $S_1^0, \dots, S_r^0$  denote the non-trivial conjugacy classes in  $\Gamma_1$  so that  $S_i$  corresponds to  $S_i^0$ . Form the element  $c = \hbar + \sum_{i=1}^r c_i \sum_{\gamma \in S_i^0} \gamma \in P \otimes \mathbb{C}\Gamma_1$ . Let  $\epsilon_i, i = 0, \dots, r$ , denote the element  $\text{tr}_i \in \mathfrak{z}^*$ . This is a basis, let  $\epsilon_0, \dots, \epsilon_r$  be the dual basis in  $\mathfrak{z}$ . Then for  $\nu$  we can take the inverse of the following map  $\mathfrak{z} \oplus \mathbb{C}\hbar \rightarrow P$ :

$$(1) \quad \hbar \mapsto \hbar, \epsilon_i \mapsto \text{tr}_{N_i} c / |\Gamma_1|, \epsilon_0 \mapsto \text{tr}_{N_0} c / |\Gamma_1| - \frac{1}{2}(c_0 + \hbar).$$

We remark that we have already seen similar formulas in Lecture 5.

Let us say a few words about the proof. Consider the group  $W_{fin}$  of the finite root system with Dynkin diagram  $\underline{Q}^{MK} \setminus \{0\}$ . We have a linear action of  $W_{fin} \times \mathbb{Z}/2\mathbb{Z}$  on  $\mathfrak{z} \oplus \mathbb{C}\hbar$  (where  $\hbar$  is invariant). There is a reduction procedure from  $\Gamma_n$  to  $\Gamma_1, \mathfrak{S}_2$ . It is based on considering completions of the algebras in consideration. The reduction shows that  $\nu$  is different from the map described by (1) by the action of an element from  $W_{fin} \times \mathbb{Z}/2\mathbb{Z}$ . But then one shows that the group of graded automorphisms of  $eHe$  that preserve  $P$  and act as identity on  $\mathbb{C}[\mathbb{C}^{2n}]^{\Gamma_n}$  is naturally identified with  $W_{fin} \times \mathbb{Z}/2\mathbb{Z}$ . So one can twist  $\nu_P$  by an element of that group and get the required isomorphism.

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