LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

IVAN LOSEV

18. Category O for rational Cherednik algebra

We begin to study the representation theory of Rational Chrednik algebras. Perhaps, the class of representations we want to start with is finite dimensional irreducible ones. Questions that we may want to ask about them is their classification and computation of dimensions.

It is unreasonable however to restrict our attention to finite dimensional representations only. The situation is similar to the representation theory of semisimple Lie algebras $\mathfrak{g}$: to classify and, especially, to compute characters of finite dimensional irreducibles one also needs to consider certain infinite dimensional representations – Verma modules. It is then reasonable to include all these modules (finite dimensional ones and Vermas) into a single category, known as the BGG (=Bernstein-Gelfand-Gelfand) category $\mathcal{O}$: the category of all finitely generated $U(\mathfrak{g})$-modules, where maximal nilpotent subalgebra acts locally nilpotently.

A similar construction can be done for a Rational Cherednik algebra (at $t \neq 0$, which reduces to $t = 1$). However, the corresponding category is, in a way, more complicated then the BGG categories $\mathcal{O}$: finite dimensional representations are no longer completely reducible, and their classification and character formulas are only known in special cases.

Our plan regarding categories $\mathcal{O}$ for RCA is as follows. Today we will give a definition, explain the highest weight structure and some related constructions. In the next four lectures we will study some functors: Bezrukavnikov-Etingof's functors, KZ functors, and, finally, Kac-Moody categorification functors.

18.1. **Some structural results on H_c.** Let us recall the definition of H_c. Let W be a complex reflection group with reflection representation $\mathfrak{h}$. Let S denote the subset of symplectic reflections in W. Pick a conjugation invariant function $c : S \to \mathbb{C}$. For $s \in S$ let $\alpha_s, \alpha_s^\vee$ denote eigenvectors for s in $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively, with non-unit eigenvalues, λ_s and λ_s^{-1}, respectively. Both vectors are defined up to a non-zero constant factor, and we partially fix the ambiguity by requiring $\langle \alpha_s, \alpha_s^\vee \rangle = 2$. Then the algebra H_c (denoted before by $H_{1,c}$) is defined as the quotient of $T(\mathfrak{h} \oplus \mathfrak{h}^*)\#W$ by the relations

$$[x, x'] = [y, y'] = 0, [y, x] = \langle y, x\rangle - \sum_{s\in S} c_s \langle \alpha_s, y\rangle\langle \alpha_s^\vee, x\rangle s, \quad x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}.$$

From the definition, we have natural homomorphisms $\mathbb{C}[\mathfrak{h}] = S(\mathfrak{h}^*), S(\mathfrak{h}), \mathbb{C}W \to H_c$. Recall that $\operatorname{gr} H_c = S(\mathfrak{h}\oplus\mathfrak{h}^*)\#W$, where the associated graded is taken with respect to the filtration defined by $\deg W = 0, \deg \mathfrak{h} \oplus \mathfrak{h}^* = 1$.

Exercise 18.1. *Use $\operatorname{gr} H_c = S(\mathfrak{h} \oplus \mathfrak{h}^*)\#W$ to show that H_c is Noetherian.*

The homomorphisms $S(\mathfrak{h}), S(\mathfrak{h}^*), \mathbb{C}W \hookrightarrow S(\mathfrak{h}\oplus\mathfrak{h}^*)\#W$ define a vector space isomorphism $S(\mathfrak{h}^*) \otimes \mathbb{C}W \otimes S(\mathfrak{h}) \xrightarrow{\sim} S(\mathfrak{h} \oplus \mathfrak{h}^*)\#W$ given by $a \otimes b \otimes c \mapsto abc$. Hence the natural map $S(\mathfrak{h}^*) \otimes \mathbb{C}W \otimes S(\mathfrak{h}) \to H_c$ is also an isomorphism of vector spaces (the map $S(\mathfrak{h}^*) \otimes \mathbb{C}W \otimes$

$S(\mathfrak{h}) \xrightarrow{\sim} S(\mathfrak{h}\oplus\mathfrak{h}^*)\#W$ is the associated graded of the map $S(\mathfrak{h}^*)\otimes\mathbb{C}W\otimes S(\mathfrak{h})\to H_c$). We will call this isomorphism a *triangular decomposition*. One should compare this with the triangular decomposition $U(\mathfrak{g})=U(\mathfrak{n}^-)\otimes U(\mathfrak{h})\otimes U(\mathfrak{n})$ for semisimple Lie algebras.

We will need a certain element $h\in H_c$ called the (deformed) Euler element. It is given by

$$h=\sum_{i=1}^n x_iy_i+\frac{1}{2}\dim\mathfrak{h}-\sum_{s\in S}\frac{2c_s}{1-\lambda_s}s. \tag{1}$$

Here y_i is a basis in $\mathfrak{h}$, x_i is the dual basis of $\mathfrak{h}^*$, and $\lambda_s\in\mathbb{C}$ is given by $s\alpha_s=\lambda_s\alpha_s$. The reason why this element is interesting (and is called Euler) is explained in the following exercise.

Exercise 18.2. *We have* $[h,x]=x,[h,w]=0,[h,y]=-y$ *for all* $x\in\mathfrak{h}^*,w\in W,y\in\mathfrak{h}$.

18.2. **Category $\mathcal{O}$.** By definition, the category $\mathcal{O}(=\mathcal{O}_c)$ (formally introduced in [GGOR]) is the full subcategory in the category of H_c-modules consisting of all modules M satisfying the following two conditions:

- M is finitely generated.
- $\mathfrak{h}$ acts on M by locally nilpotent endomorphisms.

Since H_c is Noetherian any submodule in a finitely generated module is finitely generated. It follows that $\mathcal{O}$ is an abelian category (and even a Serre subcategory in the category of all H_c-modules – it is closed under extensions).

Let us produce some examples of modules in $\mathcal{O}$. First of all, the actions of both x's and y's on any finite dimensional module are locally nilpotent, this follows from Exercise 18.2. So any finite dimensional H_c-module is on $\mathcal{O}$.

Another example is analogs of Verma modules constructed as follows. Pick a W-module E. We can view it as a $S(\mathfrak{h})\#W$-module by making $\mathfrak{h}$ act by 0. Then we set $\Delta(E)=H_c\otimes_{S(\mathfrak{h})\#W}E$. The module $\Delta(E)$ with irreducible E is called a Verma (or standard) module.

Thanks to the triangular decomposition, we have an isomorphism $H_c\cong S(\mathfrak{h}^*)\otimes(S(\mathfrak{h})\#W)$ of right $S(\mathfrak{h})\#W$-module, we get a W-equivariant isomorphism $\Delta(E)=S(\mathfrak{h}^*)\otimes E$. We have already seen such module in a special case $E=\mathrm{triv}$: it appeared as the Dunkl operator representation on $S(\mathfrak{h}^*)=\mathbb{C}[\mathfrak{h}]$, where $y\in\mathfrak{h}$ acts by the Dunkl operator

$$D_y=\partial_y-\sum_{s\in S}\frac{2c_s\langle\alpha_s,y\rangle}{(1-\lambda_s)\alpha_s}(1-s).$$

Problem 18.3. *Write an action of y on $\Delta(E)$ via a 1st order differential operator with poles.*

To check that $\Delta(E)\in\mathcal{O}$ we need to check that y acts locally nilpotently. For this (and future purposes) we will need to examine the action of h on $\Delta(E)$. On $E\subset\Delta(E)$ the element h acts by the scalar operator

$$c_E:=\frac{1}{2}\dim\mathfrak{h}-\sum_{s\in S}\frac{2}{1-\lambda_s}c_s s|_E$$

(it is scalar because it is W-equivariant, and E is irreducible). Since $[h,x]=x$, we see that the eigenvalue of h on $S^n(\mathfrak{h}^*)\otimes E$ is c_E+n. Since $[h,y]=-y$, we see that y maps $S^n(\mathfrak{h}^*)\otimes E$ to $S^{n-1}(\mathfrak{h}^*)\otimes E$ (this also can be seen directly) and so y is locally nilpotent.

Exercise 18.4. *Show that* $\mathrm{Hom}_{\mathcal{O}}(\Delta(E),M)=\mathrm{Hom}_W(E,M)$, *where* $M^{\mathfrak{h}}=\{m\in M|\mathfrak{h}m=0\}$.

Exercise 18.5. *Show that each $\Delta(E)$ has a unique irreducible quotient, denoted $L(E)$. Show that the natural inclusion $E \hookrightarrow \Delta(E)$ gives rise to an inclusion $E \hookrightarrow L(E)$. Further, show that $L(E)$ form a complete list of irreducible objects in $\mathcal{O}$.*

18.3. **Highest weight structure.** In the usual BGG category we have various triangularity properties (say, for Hom's between Verma modules or for simple constituents of Vermas). In fact, the analogous properties hold in the Cherednik category $\mathcal{O}$ as well.

We define a partial order on the set of W-irreps as follows: $E < E'$ if $c_E - c_{E'} \in \mathbb{Z}_{>0}$. The reason for introducing this ordering is as follows: if $L(E)$ appears as a composition factor in a Jordan-Hölder series of $\Delta(E')$, then either $E < E'$ or $E = E'$ and $L(E)$ is the irreducible quotient of $\Delta(E')$. Indeed, as we have seen, the eigenvalues of h in $\Delta(E')$ are $c_{E'} + n$, where $n = 0$ corresponds to the "top copy" $E' \subset \Delta(E')$. Since $L(E)$ is a composition factor of $\Delta(E')$, any eigenvalue of h in $L(E)$ is also an eigenvalue in $\Delta(E')$, i.e., of the form $c_{E'} + n$. But Exercise 18.5 implies that c_E is one of eigenvalues of h in $L(E)$. Also the eigenvalues of h in the radical of $\Delta(E)$ (the kernel of the projection $\Delta(E) \twoheadrightarrow L(E)$) are of the form $c_{E'} + n$ for $n > 0$. This implies our claim.

Exercise 18.6. *Prove that h acts locally finitely on any object in $\mathcal{O}$ and that any object in $\mathcal{O}$ has finite length. Deduce that all generalized subspaces for h are finite dimensional and that any module in $\mathcal{O}$ is finitely generated over $S(\mathfrak{h}^*)$.*

For $M \in \mathcal{O}$ we will write M_a for the generalized eigenspace for h with eigenvalue a.

Let us now give a definition of a highest weight category that formalizes various upper-triangularity properties. Let $\mathcal{O}$ be an abelian $\mathbb{C}$-linear (i.e., all Hom's are vector spaces over $\mathbb{C}$) category with finitely many simples, all objects of finite length, and sufficiently many projectives. In other words, $\mathcal{O}$ has to be equivalent to the category of finite dimensional representations of a finite dimensional algebra.

A highest weight structure on $\mathcal{O}$ is some additional data satisfying certain axioms. Let Λ be a labeling set of simples, we write $L(\lambda)$ for the simple corresponding to $\lambda \in \Lambda$ and $P(\lambda)$ for its projective cover. A *highest weight structure* on $\mathcal{O}$ is a poset structure on Λ together with so called standard objects $\Delta(\lambda)$, one for each $\lambda \in \Lambda$. These data should satisfy the following axioms.

(HW1) $\mathrm{Hom}_{\mathcal{O}}(\Delta(\lambda), \Delta(\mu)) \neq 0 \Rightarrow \lambda \leqslant \mu$.
(HW2) $\mathrm{End}_{\mathcal{O}}(\Delta(\lambda)) = \mathbb{C}$.
(HW3) There is an epimorphism $P(\lambda) \twoheadrightarrow \Delta(\lambda)$ whose kernel admits a filtration with successive quotients of the form $\Delta(\mu)$ for $\mu > \lambda$.

It is a classical fact that the BGG category $\mathcal{O}$ is a highest weight category (when we restrict to infinitesimal blocks), see, e.g., [H]. It turns out that the Cherednik category $\mathcal{O}$ is also highest weight. We have already seen that there are finitely many simples and all objects have finite length.

Exercise 18.7. *Show that (HW1) and (HW2) hold for $\mathcal{O}_c$.*

It remains to prove that there are enough projectives and that they satisfy (HW3). The proof is in several steps.

Step 0. Pick $m \in \mathbb{Z}_{\geqslant 1}$ and consider the induced module $\Delta_m(E) := H_c \otimes_{S(\mathfrak{h})\#W} (E \otimes S(\mathfrak{h})/(\mathfrak{h}^m))$. Obviously $\mathrm{Hom}_{\mathcal{O}}(\Delta_m(E), M) = \mathrm{Hom}_W(E, M^{\mathfrak{h}^m})$. Further, we have

$$\mathrm{Hom}_{H_c}(\varprojlim_m \Delta_m(E), M) = \varinjlim_m \mathrm{Hom}_W(E, M^{\mathfrak{h}^m}) = \mathrm{Hom}_W(E, \varinjlim_m M^{\mathfrak{h}^m}) = \mathrm{Hom}_W(E, M)$$

The latter defines an exact functor, and so $\varprojlim_m \Delta_m(E)$ is "kind of projective". A problem is that this object does not lie in the category $\mathcal{O}$. We will produce a projective in $\mathcal{O}$ by taking a direct summand of a "graded version" of the inverse limit.

Step 1. Consider the grading on H, where $\deg x = 1, \deg w = 0, \deg y = -1$ (the graded components are eigenspaces for $[h, \cdot]$ and the degrees are eigenvalues). Then $\Delta_m(E)$ is naturally graded with E in degree 0. We are going to establish a natural decomposition of $\Delta_m(E)$ into graded direct summands.

Let now $M = \bigoplus M^i$ be an arbitrary $\mathbb{Z}$-graded H_c-module. Since h is in degree 0, it preserves all M^i. Set $W_a(M) := \bigoplus_i M^i_{a+i}$, it is a graded submodule of M, and $M = \bigoplus_{a\in\mathbb{C}} W_a(M)$. Set $\tilde{\Delta}_m(E) = W_{c_E}(\Delta_m(E))$. Similarly, for an ungraded H_c-module M, we can consider its summands $W_{a+\mathbb{Z}}(M)$ (all generalized eigenspaces of h with e-values in $a+\mathbb{Z}$).

Of course, $\tilde{\Delta}_1(E) = \Delta(E)$ and we have a natural epimorphism $\tilde{\Delta}_{m+1}(E) \twoheadrightarrow \tilde{\Delta}_m(E)$ of graded H_c-modules. We claim that this epimorphism is iso for all sufficiently large m.

Step 2. We have an exact sequence

$$0 \to S^m(\mathfrak{h}) \otimes E \to S(\mathfrak{h})/(\mathfrak{h}^{m+1}) \otimes E \to S(\mathfrak{h})/(\mathfrak{h}^m) \otimes E \to 0.$$

Since H_c is isomorphic to $S(\mathfrak{h}^*) \otimes (S(\mathfrak{h})\#W)$ as a right $S(\mathfrak{h})\#W$-module, the induction functor $H_c \otimes_{S(\mathfrak{h})\#W} \bullet$ is exact. This yields an exact sequence of graded H_c-modules.

$$0 \to \Delta(S^m\mathfrak{h} \otimes E) \to \Delta_{m+1}(E) \to \Delta_m(E) \to 0. \tag{2}$$

The kernel $\Delta(S^m\mathfrak{h} \otimes E)$ decomposes into the direct sum of various $\Delta(E')$. The subspace $S^n(\mathfrak{h}^*) \otimes E' \subset \Delta(E')$ has degree $n - m$ and the eigenvalue of h equal to $c_{E'} + n$ (the latter follows from computations in the previous section) so will belong to the summand $W_{c_{E'}+m}(\Delta_{m+1}(E))$. For m large enough, we have $c_E \neq c_{E'} + m$ for all possible E'. Applying the exact functor W_{c_E} to exact sequence (2) we get $W_{c_E}(\Delta_{m+1}(E)) \xrightarrow{\sim} W_{c_E}(\Delta_m(E))$.

Step 3. Let $\tilde{\Delta}(E)$ denote the module $\tilde{\Delta}_m(E)$ for m sufficiently large. Let us prove that this object is projective. More precisely, we will show that $\mathrm{Hom}(\tilde{\Delta}(E), M) = \mathrm{Hom}_W(E, M_{c_E})$. Since $M \mapsto M_{c_E}$ is an exact functor, this will show that $\tilde{\Delta}(E)$ is projective.

Pick a module $M \in \mathcal{O}$. Clearly, $\mathrm{Hom}(\tilde{\Delta}(E), M) = \mathrm{Hom}(\tilde{\Delta}(E), W_{c_E+\mathbb{Z}}(M))$. So it is enough to assume that $W_{c_E+\mathbb{Z}}(M) = M$. The module M can be graded, for M^i take the generalized eigenspace of h with eigenvalue $c_E + i$.

The Hom space in $\mathcal{O}$ between two graded modules is naturally graded. Of course, $\mathrm{Hom}(\Delta_m(E), M) = \bigoplus_i \mathrm{Hom}_{\mathcal{O}}(W_{c_E-i}(\Delta_m(E)), M)$. From the definition of $W_\bullet$, the construction of grading on M and the observation that any H_c-linear homomorphism preserves the eigenspaces for h, it follows that $\mathrm{Hom}_{\mathcal{O}}(W_{c_E-i}(\Delta_m(E)), M) = \mathrm{Hom}_{\mathcal{O}}(\Delta_m(E), M)^i$. In particular, $\mathrm{Hom}_{\mathcal{O}}(\tilde{\Delta}(E), M) = \mathrm{Hom}_{\mathcal{O}}(\Delta_m(E), M)^0$ for $m \gg 0$.

The identification $\mathrm{Hom}(\Delta_m(E), M) \cong \mathrm{Hom}_W(E, M^{\mathfrak{h}^m})$ preserves the gradings, where the grading on the right hand side is induced by that on M. But also, since M is finitely generated over $\mathbb{C}[\mathfrak{h}]$, the grading on M is bounded from below. Since $\mathfrak{h}^m$ decreases degrees by m, we see that, for any given k, $(M^k)^{\mathfrak{h}^m} = M^k$ for $m \gg 0$. So $\mathrm{Hom}_{\mathcal{O}}(\Delta_m(E), M)^0 = \mathrm{Hom}_W(E, M^0) = \mathrm{Hom}_W(E, M_{c_E})$ for $m \gg 0$. This proves that $\mathrm{Hom}(\tilde{\Delta}(E), M) = \mathrm{Hom}_W(E, M_{c_E})$.

Step 4. Let us show that the module $\tilde{\Delta}(E)$ has a filtration required by (HW3). Indeed, it is filtered with successive quotients being $W_{c_E}(\Delta(S^m\mathfrak{h} \otimes E))$. The letter module splits into the sum of certain $\Delta(E')$. The top subspace $E' \subset \Delta(E') \subset W_{c_E}(\Delta(S^m\mathfrak{h} \otimes E))$ is in degree $-m$ and the eigenvalue is $c_{E'}$. So we should have $c_{E'} = c_E - m$ and hence $E < E'$.

Step 5. We are still not done. The module $\tilde{\Delta}(E)$ has a required filtration but, in general, it is not indecomposable. The claim that there is an indecomposable summand $P(E)$ of $\tilde{\Delta}(E)$ with required filtration is a consequence of the following exercise.

Exercise 18.8. *Show that if an exact sequence* $0 \to \Delta(E) \to M \to \Delta(E') \to 0$ *does not split, then* $E' < E$*. Deduce that if* $M_1 \oplus M_2$ *is* Δ*-filtered (i.e. admits a filtration whose quotients are* Δ*'s), then* M_1 *and* M_2 *are* Δ*-filtered.*

Indeed, for $P(E)$ we can take a unique indecomposable summand of $\tilde{\Delta}(E)$ that has $\Delta(E)$ (occurring in $\tilde{\Delta}(E)$ with multiplicity 1) as a filtration quotient. This is well-defined because the multiplicity of $\Delta(E)$ in a Δ-filtered object is an additive function.

References

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[H] J. Humphreys. *Representation of semisimple Lie algebras in the BGG category $\mathcal{O}$*. Available at http://www.math.umass.edu/~jeh/bgg/main.pdf