LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

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19. KZ functor I: double centralizer property

The notation $W, S, c, \alpha_s, \alpha_s^\vee$ has the same meaning as in the previous lecture. We have the category $\mathcal{O} = \mathcal{O}_c$. Sometimes we will write $\mathcal{O}_c(\mathfrak{h})$ to indicate that it is $\mathfrak{h}$ that acts locally nilpotently (we can also consider the analogous category $\mathcal{O}_c(\mathfrak{h}^*)$).

A KZ functor introduced in [GGOR] is one of the most powerful tools to study the category $\mathcal{O}_c$. It is an exact functor $\mathcal{O}_c \to \mathcal{H}_c(W)$-mod, where $\mathcal{H}_c(W)$ is the *Hecke algebra* of the complex reflection group W corresponding to the parameter c. This functor induces an equivalence of $\mathcal{O}_c/\mathcal{O}_c^{tor}$ and $\mathcal{H}_c(W)$-mod, where $\mathcal{O}_c^{tor}$ is the Serre subcategory of $\mathcal{O}_c$ consisting of all objects M in $\mathcal{O}_c$ that are *torsion* for $S(\mathfrak{h}^*)$ (i.e., any element of M is annihilated by some element of $S(\mathfrak{h}^*)$). Finally, this functor has a very nice property, it is fully faithful on projective objects (sometimes this is also called the double centralizer property).

19.1. **Localization functor.** The first ingredient to construct the KZ functor is the *localization* functor π. Consider the element $\delta \in S(\mathfrak{h}^*), \delta := \prod_{s\in S}\alpha_s$. This is a W-semiinvariant element. Recall, Exercise 13.3, that the localization $H_c[\delta^{-1}]$ makes sense and is isomorphic (via the Dunkl operator homomorphism) to $D(\mathfrak{h}^{Reg})\#W$, where $\mathfrak{h}^{Reg}$ is the principal open subset defined by δ (coinciding with the locus, where the W-action is free). So we have the localization functor $\pi: H_c\text{-mod} \to D(\mathfrak{h}^{Reg})\#W, M \mapsto D(\mathfrak{h}^{Reg})\#W \otimes_{H_c} M$. As a $\mathbb{C}[\mathfrak{h}^{Reg}]$-module, $\pi(M) = \mathbb{C}[\mathfrak{h}^{Reg}] \otimes_{\mathbb{C}[\mathfrak{h}]} M$. So, clearly, π is an exact functor.

Recall that any object $M \in \mathcal{O}$ is finitely generated over $\mathbb{C}[\mathfrak{h}]$. So $\mathrm{Loc}(M) = 0$ if and only if M is annihilated by some power of δ, i.e., is supported (as a coherent sheaf on $\mathfrak{h}$) outside $\mathfrak{h}^{Reg}$. We claim that these are all torsion modules in $\mathcal{O}$. Let us prove this. Our first observation is that $D(\mathfrak{h}^{Reg})\#W$-module $\pi(M)$ is finitely generated over $\mathbb{C}[\mathfrak{h}^{Reg}]$.

Proposition 19.1. *Let X be a smooth variety and let M be a coherent sheaf that is also a D_X-module (here D_X stands for the sheaf of differential operators on X). Then M is a vector bundle.*

Proof. It is enough to prove that the restriction of M to a formal neighborhood of any point in X is free. Consider the corresponding algebra of differential operators $D := D(\mathbb{C}[[x_1,\ldots,x_n]])$ (as a vector space, D is $\mathbb{C}[[x_1,\ldots,x_n]][\partial_1,\ldots,\partial_n]$ with obvious commutation relations). We claim that any D-module M that is finitely generated over $\mathbb{C}[[x_1,\ldots,x_n]]$ is isomorphic to the sum of several copies of $\mathbb{C}[[x_1,\ldots,x_n]]$. The module M is complete and separated in the $(x_1,\ldots,x_n)$-adic topology. We claim that any such module (not necessarily finitely generated) is the sum of several copies of $\mathbb{C}[[x_1,\ldots,x_n]]$.

The operator $p_i := \sum_{j=0}^\infty \frac{(-1)^j}{j!}x_i^j\partial_i^j$ is well-defined. It is straight-forward to check that the operators p_i pairwise commute. Also $\partial_i p_i = \sum_{j=0}^\infty \frac{(-1)^j}{j!}(jx_i^{j-1}\partial_i^j + x_i^j\partial_i^{j+1}) = 0$. Let M^∂ denote the image of $p_1\ldots p_n$. The vectors in M^∂ are annihilated by all ∂_i and they generate M modulo $(x_1,\ldots,x_n)$ (because each p_i is the identity modulo $(x_1,\ldots,x_n)$). So M^∂ generates M as a $\mathbb{C}[[x_1,\ldots,x_n]]$-module. Let us check that the natural map $\mathbb{C}[[x_1,\ldots,x_n]]\otimes M^\partial \to M$

is an isomorphism. Let $f=\sum_\alpha x^\alpha m_\alpha$ be in the kernel, here we write x^α for $x_1^{\alpha_1}\ldots x_n^{\alpha}$. We remark that the homomorphism $\mathbb{C}[[x_1,\ldots,x_n]]\otimes M^\partial\to M$ is that of D-modules. So the kernel is also a D-module. In particular, $x_i\partial_i f$ also lies in the kernel. But $x_i\partial_i x^\alpha m_\alpha = \alpha_i x^\alpha m_\alpha$. Also the kernel is closed in the $(x_1,\ldots,x_n)$-adic topology. Since with any element $\sum_\alpha x^\alpha m_\alpha$ it also contains $\sum_\alpha \alpha_i x^\alpha m_\alpha$, we see that the kernel is spanned by the monomials $x^\alpha m_\alpha$. Applying ∂^α to $x^\alpha m_\alpha$, we see that m_α lies in the kernel. But our map is tautologically injective on M^∂. So the kernel is zero. □

In particular, if the module $M\in\mathcal{O}$ is torsion, then $\pi(M)$ is torsion and hence is 0. So any torsion module in $\mathcal{O}$ is annihilated by δ^m for $m\gg 0$.

Problem 19.1. *Let M_1,M_2 be D_X-modules that are coherent sheaves. Show that*

$$\dim \operatorname{Hom}_{D_X}(M_1,M_2)<\infty.$$

19.2. **Double centralizer property.** The main result about π for today is the *double centralizer property*: the claim that π is fully faithful on projectives.

Theorem 19.2. *The functor π is fully faithful on projective objects, i.e., if P_1,P_2 are projective in $\mathcal{O}$, then the natural homomorphism $\operatorname{Hom}_{H_c}(P_1,P_2)\to\operatorname{Hom}_{D(\mathfrak{h}^{Reg})\#W}(\pi(P_1),\pi(P_2))$ is an isomorphism.*

Presently, we are in position to prove the injectivity. The surjectivity is more subtle and will be proved after some preparation.

Recall that any projective projective in $\mathcal{O}$ is Δ-filtered, i.e., admits a filtration whose successive quotients are Verma modules $\Delta(E)$. The injectivity follows from the next lemma.

Lemma 19.3. *Let $M,N\in\mathcal{O}$ and suppose that N is Δ-filtered. Then the natural homomorphism $\operatorname{Hom}_{\mathcal{O}}(M,N)\to\operatorname{Hom}_{D(\mathfrak{h}^{Reg})\#W}(\pi(M),\pi(N))$ is injective.*

Proof. Any $\Delta(E)$ is free over $S(\mathfrak{h}^*)$ and hence so is any Δ-filtered object. Being free, N is torsion-free. The equality $\pi(\varphi)=0$ for $\varphi\in\operatorname{Hom}_{\mathcal{O}}(M,N)$ is equivalent to $\pi(\operatorname{im}\varphi)=0$. But δ is a nonzero divisor on N. □

19.3. **Naive duality.** We want some conditions on M that will insure that $\operatorname{Hom}_{\mathcal{O}}(M,N)\to\operatorname{Hom}_{D(\mathfrak{h}^{Reg})\#W}(\pi(M),\pi(N))$ is surjective. This condition is kind of dual to that of Lemma 19.3. This indicates that we need to establish some kind of duality on the categories $\mathcal{O}$. Such construction is well-known for the BGG category $\mathcal{O}$ and is obtained in a similar fashion in the Cherednik case.

For a parameter $c:S\to\mathbb{C}$ define $c^\vee:S\to\mathbb{C}$ via $c^\vee(s)=c(s^{-1})$ (so that if W is a real reflection group, then $c^\vee=c$). Obviously, $(c^\vee)^\vee=c$. We will write $\mathcal{O}(\mathfrak{h}^*)$ for the category defined similarly to $\mathcal{O}$ but for $\mathfrak{h}^*$ instead of $\mathfrak{h}$. Our goal is to define an involutive contravariant equivalence $\bullet^\vee:\mathcal{O}_c\to\mathcal{O}_{c^\vee}(\mathfrak{h}^*)$.

We start by establishing an algebra isomorphism $\sigma: H_c\to H_{c^\vee}^{opp}$. On the generators, it is given by $\sigma(x)=x,\sigma(y)=-y,\sigma(w)=w^{-1}$. It is straightforward to see that $[\sigma(x),\sigma(y)]=-\sigma([x,y])$ and so σ does lift to an algebra homomorphism. Also it is clear that $\sigma^2=1$.

Now let us define the functor $\bullet^\vee$. Pick $M\in\mathcal{O}$. It admits a decomposition $M=\bigoplus_{a\in\mathbb{C}}M_a$ into generalized eigenspaces for h. Then $M^*=\operatorname{Hom}(M,\mathbb{C})$ has a natural structure of a right H-module. Consider the restricted dual $M^{(*)}=\bigoplus_a M_a^*\subset M^*$. It is stable with respect to $\mathfrak{h},W,\mathfrak{h}^*$ and so is an H-submodule. We can view $M^{(*)}$ as a left $H_{c^\vee}$-module using σ. This is the module that we are going to denote by $M^\vee$. Let us check that this module belongs to $\mathcal{O}(\mathfrak{h}^*)$. First of all, we need to check the action of $\mathfrak{h}^*$ on $M^{(*)}$ is locally nilpotent. Since

$\mathfrak{h}^* M_{a-1} \subset M_a$, it follows from $M_a^* \mathfrak{h}^* \subset M_{a-1}^*$. Using the claim that the h-eigen-values on M are bounded from below (i.e., for any eigenvalue a, the number $a - m$ is no longer an eigenvalue for $m \gg 0$), we see that $\mathfrak{h}^*$ acts locally nilpotently on $M^\vee$. The claim that $M^\vee$ is finitely generated follows from the next exercise, because one can check that the subspaces $M_a^* \subset M^\vee$ are the generalized eigenspaces for h.

Exercise 19.2. *Let M be an H_c-module with locally nilpotent action of $\mathfrak{h}$. Show that M is finitely generated iff the action of h on M is locally finite and all generalized eigen-subspaces are finite dimensional.*

It is follows from $\sigma^2 = 1$ (and the fact that taking the restricted dual is involutive) that $(M^\vee)^\vee = M$.

One application of the duality is to define so called *costandard objects* using it. Namely, we set $\nabla_c(E) := \Delta_{c^\vee}(E^*)^\vee$. This definition and the construction of duality imply that $\mathrm{Hom}_{\mathcal{O}}(M, \nabla(E)) = \mathrm{Hom}_W(M/\mathfrak{h}M, E)$. It follows that $\nabla(E)$ has simple socle equal to $L(E)$.

Problem 19.3. *Show that* $\mathrm{Ext}^i(\Delta(E), \nabla(E')) = \mathbb{C}$ *if $E = E', i = 0$, and 0 else. Moreover, show that if* $\mathrm{Ext}^1(\Delta(E), M) = 0$ *for all E, then M is ∇-filtered, i.e., admits a filtration with successive quotients $\nabla(E')$.*

Now let us study a relationship between the subcategory $\mathcal{O}^{tor}$ (of all modules in $\mathcal{O}$ that are torsion over $S(\mathfrak{h}^*)$) and the duality $\bullet^\vee$.

Lemma 19.4. *If $M \in \mathcal{O}_c^{tor}$, then $M^\vee \in \mathcal{O}(\mathfrak{h}^*)_{c^\vee}^{tor}$.*

Proof. As in the previous lecture, we may assume that all eigenvalues of h on M are congruent modulo $\mathbb{Z}$, in which case we can turn M into a graded module. Since grading is bounded from below, we can shift it and assume that M is positively graded, $M = \bigoplus_{i \geqslant 0} M^i$. Then $M^\vee$ is naturally negatively graded, $M^\vee = \bigoplus_{i \leqslant 0} (M^\vee)^i$ with $(M^\vee)^i = (M^{-i})^*$. The condition that M is $S(\mathfrak{h})$-torsion is equivalent to $\lim_{i \to +\infty} i^{1-n} \dim M_i = 0$, where $n = \dim \mathfrak{h}$. The condition that $M^\vee$ is $S(\mathfrak{h}^*)$-torsion is equivalent to $\lim_{i \to -\infty} i^{1-n} \dim (M^\vee)^i = 0$. Since $\dim (M^\vee)^i = \dim M^{-i}$, we are done. ☐

Lemma 19.5. *Let M be ∇-filtered. Then* $\mathrm{Hom}_{\mathcal{O}}(M, N) \twoheadrightarrow \mathrm{Hom}_{D(\mathfrak{h}^{Reg})\#W}(\pi(M), \pi(N))$.

Proof. By the construction,

$$\mathrm{Hom}_{D(\mathfrak{h}^{Reg})\#W}(\pi(M), M') = \mathrm{Hom}_{H_c}(M, M')$$

and so what we need to show is that

$$\mathrm{Hom}_{H_c}(M, N) \to \mathrm{Hom}_{D(\mathfrak{h}^{Reg})\#W}(\pi(M), \pi(N)) = \mathrm{Hom}_{H_c}(M, \pi(N))$$

is surjective. This will follow if we check that $\mathrm{Hom}_{H_c}(M, \pi(N)/N) = 0$. For any vector $v \in \pi(N)$ there is $m > 0$ such that $\delta^m v \in N$. So $\pi(N)/N$ is torsion. By Lemma 19.4, any ∇-filtered object has no torsion quotients. So $\mathrm{Hom}_{H_c}(M, \pi(N)/N) = 0$. ☐

An object in $\mathcal{O}$ is called *tilting* if it is both Δ- and ∇-filtered (the filtrations are different, in general).

Corollary 19.6. *If M, N are tilting, then* $\mathrm{Hom}_{\mathcal{O}}(M, N) \xrightarrow{\sim} \mathrm{Hom}_{D(\mathfrak{h}^{Reg})\#W}(\pi(M), \pi(N))$.

19.4. **Homological/Ringel duality.** Generally, projectives are not tilting. Indeed, any simple module occurs in the head of some projective, while only torsion-free simple can occur in the head of a tilting. Projectives and tiltings are again kind of dual (but not via the naive duality, it sends tiltings to tiltings, and projectives to injectives). To make this statement precise we note that, according to Problem 19.3, a Δ-filtered object M is tilting iff $\operatorname{Ext}^1(\Delta(E), M) = 0$ for all E.

Problem 19.4. *A Δ-filtered object M is projective iff $\operatorname{Ext}^1(M, \Delta(E)) = 0$ for all E.*

So what we need is an exact contravariant duality between the categories of Δ-filtered objects (these are *exact categories*, in particular, the notions of an exact sequence and so of Ext^1 make sense), maybe with different parameters, that is compatible with the localization functors. We note that such duality cannot be exact on the whole category $\mathcal{O}_c$ (then it would map projectives to injectives, not to tiltings). Such duality exists for all highest weight categories and is known as a *Ringel* duality. In our particular case, this duality is a standard homological duality that makes sense for all rings.

Namely, let A be an associative algebra with unit. Then we have the functor $\operatorname{Hom}_A(\bullet, A)$ from the category of left A-modules to the category of right A-modules. This functor behaves well on free or, more generally, projective A-modules (in particular, it is involutive) has no other good properties (for example, it often sends a module to 0). To remedy this, one considers the derived functor $\operatorname{RHom}(\bullet, A)$ from the category of finitely generated left A-modules to the derived category of right A-modules. By definition, this functor maps an A-module M to the complex $\operatorname{Hom}_A(F_0, A) \to \operatorname{Hom}_A(F_1, A) \to \ldots$, where $\ldots \to F_1 \to F_0$ is a free resolution of M and the A^{opp}-action on the complex is given by the action of A on A from the right. In particular, if A has finite homological dimension, then the functor induces a contravariant equivalence $D : D^b(A) \to D^b(A^{opp})$, where $D^b(A)$ stands for the bounded (=of complexes of finite length) derived category of the category of left A-modules. We remark that $D^2 = \mathrm{id}$.

The algebra H_c has finite homological dimension, by Problem 9.6. We can view right H_c-modules as left $H_{c^\vee}$-modules and so we get a contravariant equivalence $D : D^b(H_c) \to D^b(H_{c^\vee})$ with $D^2 = 1$. By the construction, D commutes with the localization and so $D \circ \pi_c = \pi_{c^\vee} \circ D$.

Let us compute $D(\Delta_c(E))$.

Lemma 19.7. *We have $H^i(D(\Delta_c(E))) = 0$ if $i \neq n$ and $H^n(D(\Delta_c(E))) = \Delta_{c^\vee}(E')$ for certain E'.*

Proof. Of course, $H^i(D(\Delta_c(E))) = \operatorname{Ext}^i_{H_c}(\Delta_c(E), H_c)$. We have $\Delta_c(E) = H_c \otimes_{S(\mathfrak{h})\#W} (E)$. Since $H_c \otimes_{S(\mathfrak{h})\#W} \bullet$ is an exact functor, and its right adjoint functor (the restriction from H_c to $S(\mathfrak{h})\#W$) is also exact, we see that $\operatorname{Ext}^i_{H_c}(\Delta_c(E), H_c) = \operatorname{Ext}^i_{S(\mathfrak{h})\#W}(E, H_c)$. By the triangular decomposition (written in the opposite order), H_c is a free left $S(\mathfrak{h})\#W$-module. So $\operatorname{Ext}^i_{S(\mathfrak{h})\#W}(E, H_c) = \operatorname{Ext}^i_{S(\mathfrak{h})\#W}(E, S(\mathfrak{h})\#W) \otimes_{S(\mathfrak{h})\#W} H_c$. Since a $S(\mathfrak{h})\#W$-linear map is the same as a W-equivariant and $S(\mathfrak{h})$-linear map, we see that

$$\operatorname{Ext}^i_{S(\mathfrak{h})\#W}(E, S(\mathfrak{h})\#W) = \operatorname{Ext}^i_{S(\mathfrak{h})}(E, S(\mathfrak{h})\#W)^W = \left(\operatorname{Hom}(E, \mathbb{C}W) \otimes \operatorname{Ext}^i_{S(\mathfrak{h})}(\mathbb{C}, S(\mathfrak{h}))\right)^W$$

The $S(\mathfrak{h})$-module $\operatorname{Ext}^i_{S(\mathfrak{h})}(\mathbb{C}, S(\mathfrak{h}))$ is 0 if $i \neq n$ and is $\bigwedge^n \mathfrak{h}^*$ if $i = n$. To see this one uses the Koszul resolution of the $S(\mathfrak{h})$-module $\mathbb{C}$. We conclude that $\operatorname{Ext}^i_{H_c}(\Delta_c(E), H_c) = 0$ for $i \neq n$ and $\operatorname{Ext}^i_{H_c}(\Delta_c(E), H_c) = E'' \otimes_{S(\mathfrak{h})\#W} H_c$, where $E'' = [\operatorname{Hom}(E, \mathbb{C}W) \otimes \bigwedge^n \mathfrak{h}^*]^W$ is an irreducible right W-module. So we have $H^i(D(\Delta_c(E))) = 0$ if $i \neq n$ and $H^n(D(\Delta_c(E))) =$

$\Delta_{c^\vee}(E')$ for an irreducible left W-module E' (equal to $\bigwedge^n \mathfrak{h}^* \otimes E^*$, but we will not need this). $\square$

From now on, we replace D with $D[n]$, so that $D(\Delta_c(E))$ is concentrated in the homological degree 0. Since an abelian category is a full subcategory of its derived category (of all complexes with cohomology only in degree 0), we conclude that we have a contravariant equivalence $D : \mathcal{O}_c^\Delta \to \mathcal{O}_{c^\vee}^\Delta$ (the superscript Δ stands for the category of Δ-filtered objects) that maps standards to standards. By problems 1 and 2, this equivalence has to map tiltings to projectives and vice versa.

Now we are in position to finish the proof of Theorem 19.2.

Proof of Theorem 19.2. Let P_1, P_2 be projectives in $\mathcal{O}_c$. Then, since $\pi \circ D_{H_c} = D_{D(\mathfrak{h}^{Reg})\#W} \circ \pi$, we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{O}_c}(P_1, P_2) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{O}_{c^\vee}}(D(P_2), D(P_1)) \arrow[d] \\
\mathrm{Hom}_{D(\mathfrak{h}^{Reg})\#W}(\pi(P_1), \pi(P_2)) \arrow[r] & \mathrm{Hom}_{D(\mathfrak{h}^{Reg})\#W}(\pi\, D(P_2), \pi\, D(P_1))
\end{tikzcd}$$

The horizontal arrows in this diagram are induced by D, which is an equivalence of derived categories, and so are isomorphisms. The right vertical arrow is an isomorphism by Corollary 19.6. So the left vertical arrow is an isomorphism. $\square$

19.5. **Quotient property.** We claim that the functor π admits a right adjoint π^* that maps a finitely generated $D(\mathfrak{h}^{Reg})\#W$-module N to the sum of all its submodules lying in $\mathcal{O}$. The natural isomorphism $\mathrm{Hom}_{H_c}(M, \pi^*(N)) \cong \mathrm{Hom}_{H_c}(\pi(M), N) = \mathrm{Hom}_{D(\mathfrak{h}^{Reg})\#W}(\pi(M), N)$ is clear, the only thing we need to check is that $\pi^*(N)$ is finitely generated. By Problem 19.1, $\dim \mathrm{Hom}_{D(\mathfrak{h}^{Reg})\#W}(\pi(M), N) < \infty$. Applying this to the projective $P(E)$, we see that $\dim \mathrm{Hom}_{H_c}(P(E), \pi^*(N)) < \infty$. So $\pi^*(N)$ has finite length and hence is in $\mathcal{O}$.

Since $\pi(M)/M$ is torsion, we see that $\pi^*\,\pi(M)/M$ is torsion and so $\pi\,\pi^*\,\pi(M) = \pi(M)$. So for $N \in \operatorname{im}\pi$ we have $\pi\pi^*(N) = N$. It follows that $\operatorname{im}\pi$ is isomorphic to the quotient category $\mathcal{O}/\mathcal{O}^{tor}$ and π is the quotient functor (the quotient functor is one satisfying a universality property for all functors that kills a given Serre subcategory). We will describe the image of π in the following lecture.

Since $\mathcal{O}$ is equivalent to the category of finite dimensional representations of some finite dimensional algebra, say A, the functor π can be described as follows. Let e be an idempotent in A corresponding to the projective that covers the torsion-free simples. Then $\operatorname{im}\pi$ is the category of modules over eAe and the functor π is $M \mapsto eM$.

The following problem shows that π is fully faithful on injectives and has left adjoint.

Problem 19.5.
(1) *Show that the double centralizer property is equivalent to* $\mathrm{Ext}^1(M, P) = 0$ *for any projective* P *and* $M \in \mathcal{O}^{tor}$.
(2) *Use the naive duality to show that* π *is fully faithful on injectives.*
(3) *Show that* π *has left adjoint* $\pi^!$ *and that* $\pi \circ \pi^!$ *is the identity on the image of* π.

References

[GGOR] V. Ginzburg, N. Guay, E. Opdam and R. Rouquier, *On the category $\mathcal{O}$ for rational Cherednik algebras*, Invent. Math., **154** (2003), 617-651.