SRA. Lec 21.

0) Reminder.

1) Compl. of proof of Thm

2) Induction & Restrictions for $\mathcal{H}_q(W)$

3) Isomorphism of completions for RCA

0) $M\in\mathcal{O}_c \rightsquigarrow$ Sheaf $\pi(M)\in Coh(\mathfrak{h}^{W,Reg}) \rightsquigarrow M'\in Coh(\mathfrak{h}^{Reg}/W)$

- accounts for $S(\mathfrak{h}^*)$ & $\mathbb{C}W$-actions,

$\mathfrak{h}$-action $\rightsquigarrow$ flat conn. ∇ on $M' \rightsquigarrow B_W:=\pi_1(\mathfrak{h}^{Reg}/W,x)\curvearrowright M'_x$ (nearby fibers of M are canon. ident.) $B_W=\langle T_H,\ H:=\ker\alpha_s\rangle$/rel-ns

$H\rightsquigarrow W_H$ w. $\ell_H=|W_H|$, $c: S\to W \rightsquigarrow q_{H,i}\in\mathbb{C}^\times\setminus\{0\}$, $q_{H,0}=1$.

$\rightsquigarrow \mathcal{H}_c(W)=\mathbb{C}B_W/\left(\prod_{j=0}^{\ell_H-1}(T_H-q_{H,j})\right)$ $\quad \mathcal{H}_0(W)=\mathbb{C}W$, Hypoth. $\dim\mathcal{H}_c(W)=|W|$

$\mathcal{H}_c(W)\curvearrowright M_x$. $\quad \dim M_x$ = gen. rk of M as $S(\mathfrak{h}^*)$-module

$KZ: M\mapsto M_x$, $KZ=\mathrm{Hom}_{\mathcal{O}}(P_{KZ},\bullet)$, P_{KZ} - proj. w.

$\varphi: \mathcal{H}_c(W)\longrightarrow \mathrm{End}_{\mathcal{O}}(P_{KZ})$. Have seen φ is surjective

Remains $\dim \underbrace{\mathrm{End}_{\mathcal{O}}(P_{KZ})}_{B:=}=|W|$

1) mult of $P(E)$ in $P_{KZ}=\dim_{\mathcal{O}}(P_{KZ},L(E))=\dim KZ(L(E))$

$\dim B=\sum_{E,E'}\dim KZ(L(E))\,KZ(L(E'))\dim \mathrm{Hom}_{\mathcal{O}}(P(E),P(E'))$

$\dim\mathrm{Hom}_{\mathcal{O}}(P(E),P(E'))=[P(E'):L(E)]=\sum_{E''}[P(E'):\Delta(E'')][\Delta(E''):L(E)]$

Δ-filt

$\dim B=\sum_{E''}\Big(\underbrace{\sum_E \dim KZ(L(E))[\Delta(E''):L(E)]}_{=\dim KZ(\Delta(E''))\text{ b/c }KZ\text{ is exact}\ =\dim E''}\Big)\Big(\sum_{E'}\dim KZ(L(E'))[P(E'):\Delta(E'')]\Big)$

claim: $[\nabla(E''):L(E')]$

Reason: M – Δ-filt.

$[M:\Delta(E'')]=\mathrm{Hom}_{\mathcal{O}}(M,\nabla(E''))$ (Prob. 19.2).

$M=P(E')$: $\quad =\mathrm{Hom}_{\mathcal{O}}(P(E'),\nabla(E''))=[\nabla(E''):L(E')]$

So 2nd $(\ldots)=\dim KZ(\nabla(E''))$

Claim: gen. rk $\nabla(E'')$ = gen. rk $\Delta(E'')$

Reason: $M=\bigoplus_i M^i$ - graded / $\mathbb{C}[x_1..x_n]$, then gen. rk $M=\lim_{i\to+\infty} i^{1-n}\dim M_i$

Then compare w. Lem 19.4.

So $\dim B = \sum_{E''} (\dim E'')^2 = |W|$.

2) Classic: $H \subset G$ - fin. grps: Res_G^H: G-Rep $\to$ H-Rep,

Ind_H^G: H-Rep $\to$ G-Rep: $N \mapsto \mathbb{C}G \otimes_{\mathbb{C}H} N$

$\overset{\wr\|}{\mathrm{Coind}}{}_H^G$: H-Rep $\to$ G-Rep $N \mapsto \mathrm{Hom}_{\mathbb{C}H}(\mathbb{C}G, N)$

So Res_G^H, Ind_H^G - exact biadjoint functors

Goal: version for Hecke algebras:

$b \in \mathfrak{h} \rightsquigarrow \underline{W} := W_b \subset W$ - also gen. by refl-ns

refl-n rep-n $\mathfrak{h}_{\underline{W}}$ = unique $\underline{W}$-stab. compl. in $\mathfrak{h}^{\underline{W}}$.

compl. refl-ns in $\underline{W} = S \cap \underline{W}$: $c: S \to \mathbb{C} \rightsquigarrow c: S \cap \underline{W} \to \mathbb{C}$

$\rightsquigarrow \mathcal{H}_c(\underline{W})$

Lem: $\mathcal{H}_c(\underline{W}) \hookrightarrow \mathcal{H}_c(W)$

Sketch of proof: $B_{\underline{W}} \hookrightarrow B_W$: $\exists$ neigh-d of Wb = disc $\times$ neigh-d of 0 in $\mathfrak{h}_{\underline{W}}/\underline{W}$ so loop in $\mathfrak{h}_{\underline{W}}^{reg}/\underline{W} \rightsquigarrow$ loop in $\mathfrak{h}^{reg}/W$: $T_{\underline{H}} \mapsto T_H \rightsquigarrow \mathcal{H}_c(\underline{W}) \to \mathcal{H}_c(W)$ - injective $\square$

Fact (enhancement of hypoth.) $\mathcal{H}_c(W)$ is free left/right module / $\mathcal{H}_c(\underline{W})$

(Will elaborate for $W = G(\ell, 1, n)$ later)

Lem: $\rightsquigarrow$ $\mathrm{Res}_{\underline{W}}^W: \mathcal{H}_c(W)\text{-mod} \to \mathcal{H}_c(\underline{W})$-mod, $\mathrm{Ind}_{\underline{W}}^W$, $\mathrm{Coind}_{\underline{W}}^W: \mathcal{H}_c(\underline{W})$-mod $\to \mathcal{H}_c(W)$-mod

Hypothesis: $\mathcal{H}_c(W)$ is symmetric i.e. $\mathcal{H}_c(W) \underset{\mathcal{H}_c(W)\text{-bimod}}{\simeq} \mathcal{H}_c(W)^* \Longrightarrow \mathrm{Ind}_{\underline{W}}^W \simeq \mathrm{Coind}_{\underline{W}}^W$

- holds for Weyl groups & all $G(\ell, r, n)$

- will elaborate for $G(\ell, 1, n)$ later

Goal: similar functors for H_c: $\mathrm{Res}_{\underline{W}}^W: \mathcal{O}_c(W, \mathfrak{h}) \to \mathcal{O}_c(\underline{W}, \mathfrak{h})$

$\mathrm{Ind}_{\underline{W}}^W: \mathcal{O}_c(\underline{W}, \mathfrak{h}) \longrightarrow \mathcal{O}_c(W, \mathfrak{h})$

- exact, biadjoint & satisfying $\underset{\text{Cherednik}}{KZ \circ \mathrm{Res}_{\underline{W}}^W} = \underset{\text{Hecke}}{\mathrm{Res}_{\underline{W}}^W} \circ KZ \underset{adj}{\Leftrightarrow} KZ \circ \mathrm{Ind}_{\underline{W}}^W = \mathrm{Ind}_{\underline{W}}^W \circ KZ$.

Problem: $H_c(\underline{W}, \mathfrak{h})$ is not subalg. in $H_c(W, \mathfrak{h})$ (and this wouldn't help anyway)

Fix: some isomorphism of completions (Bezrukavnikov-Etingof)

3) $b\in\mathfrak h \rightsquigarrow b:\mathbb C[\mathfrak h]\to\mathbb C \rightsquigarrow$ restr to $b:\mathbb C[\mathfrak h/W], \mathbb C[\mathfrak h/\underline W]\to\mathbb C$ ($\underline W = W_b$)

$\mathbb C[\mathfrak h]^{\wedge b}$, $\mathbb C[\mathfrak h/W]^{\wedge b}$, $\mathbb C[\mathfrak h/\underline W]^{\wedge b}$. since $\mathfrak h/\underline W\to\mathfrak h/W$ is etale at $\underline W b$:

$$\mathbb C[\mathfrak h/W]^{\wedge b} = \mathbb C[\mathfrak h/\underline W]^{\wedge b}$$

Q: How about $\mathbb C[\mathfrak h]\#W$ & $\mathbb C[\mathfrak h]\#\underline W$ or more precisely

$$(\mathbb C[\mathfrak h]\#W)^{\wedge b} := \mathbb C[\mathfrak h/W]^{\wedge b}\otimes_{\mathbb C[\mathfrak h/W]}\mathbb C[\mathfrak h]\#W = \mathbb C[\mathfrak h]^{\wedge Wb}\#W \leftarrow \Big(\bigoplus_{v\in Wb}\mathbb C[\mathfrak h]^{\wedge v}\Big)\#W$$

and $(\mathbb C[\mathfrak h]\#\underline W)^{\wedge b} = \mathbb C[\mathfrak h/\underline W]^{\wedge b}\otimes_{\mathbb C[\mathfrak h/\underline W]}\mathbb C[\mathfrak h]\#\underline W = \mathbb C[\mathfrak h]^{\wedge b}\#\underline W$ - not as unital subalg.

Rather: $(\mathbb C[\mathfrak h]\#W)^{\wedge b}\simeq \mathrm{Mat}_{|W/\underline W|}(\mathbb C[\mathfrak h]^{\wedge b}\#\underline W)$

More invariant: $H\subset G$ - fin. groups, $A\supset\mathbb CH$ - ass. alg. w. 1 $\rightsquigarrow \mathrm{Hom}_H(\mathbb CG, A)$ ~~right A-module~~

$= \{\varphi:\mathbb CG\to A \mid \varphi(hg)=h\varphi(g)\}$

free right A-module trivialized by choice of el-ts in all Hg.

$\rightsquigarrow Z(G,H,A) := \mathrm{End}_{A^{opp}}(\mathrm{Hom}_H(\mathbb CG,A))\simeq \mathrm{Mat}_{|G/H|}(A)$

Lem: $(\mathbb C[\mathfrak h]\#W)^{\wedge b}\simeq Z(W,\underline W,\mathbb C[\mathfrak h]^{\wedge b}\#\underline W)$

Proof: Need compat. homom-sms $\mathbb C[\mathfrak h],\mathbb CW\to Z(W,\underline W,\mathbb C[\mathfrak h]\#\underline W)$

General: $G\to Z(G,H,A)$ - need $G\curvearrowright\mathrm{Hom}_H(\mathbb CG,A)$ commuting w. A

$g.\varphi(h)=\varphi(hg)$: need $f\in\mathbb C[\mathfrak h]$, $\varphi\in\mathrm{Hom}_H(\mathbb CG,A)\rightsquigarrow f\cdot\varphi$

$(f\cdot\varphi)(w) = ({}^{w}f)\cdot\varphi(w)$

Problem ~~Exercise~~ *: gives $\mathbb C[\mathfrak h]\#W\longrightarrow Z(W,\underline W,\mathbb C[\mathfrak h]\#\underline W)$ that lifts to iso

$$(\mathbb C[\mathfrak h]\#W)^{\wedge b}\xrightarrow{\sim} Z(W,\underline W,\mathbb C[\mathfrak h]^{\wedge b}\#\underline W).$$

Finally on the level of RCA: $H_c = H_c(W,\mathfrak h)$, $\underline H_c = H_c(\underline W,\mathfrak h)$.

$H_c^{\wedge b} := \mathbb C[\mathfrak h/W]^{\wedge b}\otimes_{\mathbb C[\mathfrak h/W]} H_c$ - algebra w. multiplication extended from H_c by continuity: reason $\mathfrak m_b\subset\mathbb C[\mathfrak h/W]$ - max ideal $\Rightarrow$

$[y,\mathfrak m_b^k]\subset\mathfrak m_b^{k-1}\cdot\mathbb C[\mathfrak h]\#W$ $\forall y\in\mathfrak h$.

$\underline H_c^{\wedge b} := \mathbb C[\mathfrak h/\underline W]^{\wedge b}\otimes_{\mathbb C[\mathfrak h/\underline W]}\underline H_c$.

Thm (Bezrukavnikov-Etingof): $\exists!$ iso $\theta: H_c^{\wedge b}\xrightarrow{\sim} Z(W,\underline W,\underline H_c^{\wedge b})$ ~~given on~~ restricting to isom $(\mathbb C[\mathfrak h]\#W)^{\wedge b}\xrightarrow{\sim} Z(W,\underline W,\mathbb C[\mathfrak h]^{\wedge b}\#\underline W)$

and $y\longmapsto\theta(y)$ s.t. $[\theta(y)\varphi](w) = (wy)\varphi(w) + \sum_{s\in S\setminus\underline W}\frac{2c_s}{1-\lambda_s}\frac{\langle\alpha_s, wy\rangle}{\alpha_s}\cdot(\varphi(sw)-\varphi(w))$

Problem: prove thm.

Addit. summand in $\theta(y)$ is really a part of Dunkl operator "outside of $\underline{W}$" – that's how they obtain the formula

Rem: $H^{\wedge_b} \xrightarrow{\sim}$ ~~[illegible]~~ $D(\mathfrak{h}^{\underline{W}})^{\wedge_b} \hat{\otimes} \underline{H}_c(\underline{W}, \mathfrak{h}_{\underline{W}})^{\wedge_0}$

allow some infinite sums

$\downarrow \wr$ ← shift by b

$\underline{H}^{\wedge_0}$ possible b/c b is $\underline{W}$-equiv.

To produce $Res^W_{\underline{W}}$, $Ind^W_{\underline{W}}$ (functors depending on choice of b) introduce intermediate category $\mathcal{O}_c^{\wedge_b} = \{$modules$/ H_c^{\wedge_b}$ fin. gen over $\mathbb{C}[\mathfrak{h}]^{\wedge Wb}\}$

Next time: equivalence $\iota: \mathcal{O}_c^{\wedge_b} \xrightarrow{\sim} \mathcal{O}_c(\underline{W}, \mathfrak{h}_{\underline{W}})$

completion functor: $\bullet^{\wedge_b}: \mathcal{O}_c \to \mathcal{O}_c^{\wedge_b}: M \mapsto \mathbb{C}[\mathfrak{h}/W]^{\wedge_b} \otimes_{\mathbb{C}[\mathfrak{h}/W]} M$

right. Its [redacted] adjoint: $E_0: \mathcal{O}_c^{\wedge_b} \to \mathcal{O}_c$: $N \mapsto$ gen. e-space of $\mathfrak{h}$ w. e-value 0 in N.

$Res^W_{\underline{W}} = \iota \circ (\bullet^{\wedge_b})$, $Ind^W_{\underline{W}} = E_0 \circ \iota^{-1}$ – not clear so far why image lies in $\mathcal{O}_c$ (why fin. generated)