SRA Lecture 22
Ind & Res: continued

0) Reminder.
1) Equivalence $\mathcal{O}_c^{\wedge_b} \xrightarrow{\sim} \underline{\mathcal{O}}_c^+$
2) Exactness of Ind
3) Properties of Res & Ind

0) Reminder: $b \in \mathfrak{h} \rightsquigarrow \underline{W} = W_b$, $H_c^{\wedge_b} = \mathbb{C}[\mathfrak{h}/W]^{\wedge_b} \otimes_{\mathbb{C}[\mathfrak{h}/W]} H_c$

$\underline{H}_c = H_c(\mathfrak{h}, \underline{W})$, $\underline{H}_c^{\wedge_b} = \mathbb{C}[\mathfrak{h}/\underline{W}]^{\wedge_b} \otimes_{\mathbb{C}[\mathfrak{h}/\underline{W}]} \underline{H}_c$.

$\rightsquigarrow Z(W, \underline{W}, \underline{H}_c^{\wedge_b}) = End_{\underline{H}_c^{\wedge_b}}(Hom_{\underline{W}}(\mathbb{C}W, \underline{H}_c^{\wedge_b}))$

Have iso $H_c^{\wedge_b} \xrightarrow[\theta]{\sim} Z(W, \underline{W}, \underline{H}_c^{\wedge_b})$

$[\theta(w)\varphi](w') = \varphi(w'w)$ $\quad \varphi \in Hom_{\underline{W}}(\mathbb{C}W, \underline{H}_c^{\wedge_b})$, $w' \in W$, $w \in W$,

$[\theta(x)\varphi](w') = w'x \cdot \varphi(w')$ $\quad x \in W$

$[\theta(y)\varphi](w') =$ some formula: $w'y \cdot \varphi(w') +$ correction

$\mathcal{O}_c^{\wedge_b} := \{M \in H_c^{\wedge_b}\text{-mod} \mid M \text{ is fin. gen} / \mathbb{C}[\mathfrak{h}/W]^{\wedge_b}\}$

Functors: $\bullet^{\wedge_b}: \mathcal{O}_c \to \mathcal{O}_c^{\wedge_b}: M \mapsto \mathbb{C}[\mathfrak{h}/W]^{\wedge_b} \otimes_{\mathbb{C}[\mathfrak{h}/W]} M$ – exact

$E_0: \mathcal{O}_c^{\wedge_b} \to \tilde{\mathcal{O}}_c = \{M \in H_c\text{-Mod} \mid \mathfrak{h} \text{ acts loc. nilp}\}: N \mapsto$ gen e-space for $\mathfrak{h}$ w. e-value 0.

$+$ will construct $\iota: \mathcal{O}_c^{\wedge_b} \xrightarrow{\sim} \underline{\mathcal{O}}_c^+ := \mathcal{O}_c(\mathfrak{h}_{\underline{W}}, \underline{W})$

$\rightsquigarrow Res_{\underline{W}}^W: \mathcal{O}_c \to \underline{\mathcal{O}}_c^+: M \mapsto \iota(M^{\wedge_b})$

$Ind_{\underline{W}}^W: \underline{\mathcal{O}}_c^+ \to \tilde{\mathcal{O}}_c: N \mapsto E_0(\iota^{-1}(N))$

$$\mathcal{O}_c^{\wedge_b} \xrightarrow{\sim} \mathcal{O}^{\wedge_b}_{Z(W,\underline{W},\underline{H}_c^{\wedge_b})} \xrightarrow{\sim} \underline{\mathcal{O}}_c^{\wedge_b} \xrightarrow{\sim} \underline{\mathcal{O}}_c^{+\wedge_0} \xrightarrow{\sim} \underline{\mathcal{O}}_c^+$$

1) a) Equiv. $H_c^{\wedge_b}\text{-mod} \xrightarrow{\sim} Z(W, \underline{W}, \underline{H}_c^{\wedge_b})\text{-mod}: \theta_*$.

maps $\mathcal{O}_c^{\wedge_b}$ to $\{M \in Z(W, \underline{W}, \underline{H}_c^{\wedge_b})\text{-mod} \mid M \text{ is fin. gen} / \mathbb{C}[\mathfrak{h}/\underline{W}]^{\wedge_b}\} = \mathcal{O}^{\wedge_b}_{Z(W,\underline{W},\underline{H}_c^{\wedge_b})}$

b) Equiv. $Z(W, \underline{W}, \underline{H}_c^{\wedge_b})\text{-mod} \xrightarrow{\sim} \underline{H}_c^{\wedge_b}\text{-mod}$:

$e(\underline{W}) \in \underbrace{Z(W, \underline{W}, \underline{H}_c^{\wedge_b})}_{\text{matrix alg. } / \underline{H}_c^{\wedge_b}}$, $\quad e(\underline{W})\varphi(w) = \begin{cases} \varphi(w), & w \in \underline{W} \\ 0, & \text{else} \end{cases}$ $\quad M \mapsto e(\underline{W})M$

matrix unit

c) Equivalence $\underline{\mathcal{O}}_c^{\wedge_b} \xrightarrow{\sim} \underline{\mathcal{O}}_c^{+\wedge_0}$

$\underline{H}_c \xrightarrow{\sim} \mathcal{D}(\mathfrak{h}^W)\otimes_{\mathbb{C}} \underline{H}_c^+ \to \underline{H}_c^{\wedge_b} \xrightarrow{\sim} \mathcal{D}(\mathfrak{h}^W)^{\wedge_b}\hat{\otimes}\, \underline{H}_c^{+\wedge_0}$

allow some infinite sums

$M\in\underline{\mathcal{O}}_c^{\wedge_b}$ is compl. & sep. in $\underline{m}_b$-adic topol., where $\underline{m}_b$ is max. ideal of b in $\mathbb{C}[\mathfrak{h}/\underline{W}]$ $\Rightarrow$ compl. & sep in $\hat{\underline{m}}_b$-adic topol., $\hat{\underline{m}}_b \in \mathbb{C}[\mathfrak{h}^{\underline{W}}]$

$\| \quad \mathbb{C}[\mathfrak{h}^{\underline{W}}]\otimes \mathbb{C}[\mathfrak{h}_{\underline{W}}/\underline{W}]$

$\Downarrow$ Prop. 19.1

$$M \simeq \mathbb{C}[\mathfrak{h}^W]^{\wedge_b}\hat{\otimes} M'$$

M' = Annih. of $\mathfrak{h}^W$, obj. in $\underline{\mathcal{O}}^{+\wedge_0}$.

Equiv. $M\mapsto M'$ (inverse $M'\mapsto \mathbb{C}[\mathfrak{h}^W]^{\wedge_b}\hat{\otimes} M'$)

d) Equiv. $\underline{\mathcal{O}}_c^{+\wedge_0}\xrightarrow{\sim}\underline{\mathcal{O}}_c^{+}$

$\longleftarrow$: $N\mapsto N^{\wedge_0}$

$\longrightarrow$: $M\mapsto$ finite vectors for Euler element.

<u>Problem 1</u>: Check these two are (quasi) inverse to each other.

Corollary of existence of i: All Hom spaces in $\mathcal{O}_c^{\wedge_b}$ are fin. dim $\Rightarrow$ $E_0(M)$ has fin. length $\forall\, M\in\mathcal{O}_c^{\wedge_b}$ (b/c $[\mathrm{Hom}_{H_c}(M', E_0(M)) = \mathrm{Hom}_{H_c}(M'^{\wedge_b}, M)]$) $\Rightarrow E_0:\mathcal{O}_c^{\wedge_b}\to\mathcal{O}_c$ is right adjoint to $\bullet^{\wedge_b}$.

2) Exactness of Ind $\Leftrightarrow$ of E_0 | Recall $M\mapsto M^\vee$ = fin. vectors in $M^* = \mathrm{Hom}(M,\mathbb{C})$

$\bullet^*: M\to \mathrm{Hom}(M,\mathbb{C})$: $\mathcal{O}_c(\mathfrak{h},W)\xrightarrow{\sim}\mathcal{O}_{c^\vee}(\mathfrak{h}^*,W)^{\wedge_0}$

$\bullet^*: N\to \mathrm{Hom}_{cont}(N,\mathbb{C}) = \{\varphi: N\to\mathbb{C},\ \varphi((\mathfrak{h})^{n}N)=0,\ n\gg 0\}$

: $\mathcal{O}_{c^\vee}(\mathfrak{h}^*,W)^{\wedge_0}\xrightarrow{\sim}\mathcal{O}_c(\mathfrak{h},W)$

$\bullet^*: N\to\mathrm{Hom}_{cont}(N,\mathbb{C})$: $\mathcal{O}_c(\mathfrak{h},W)^{\wedge_b}\xrightarrow{\sim}\mathcal{O}_{c^\vee}(\mathfrak{h}^*,W)_b$ =

= full subcat. of $H_{c^\vee}$-mod that are

- finite length ~~• fin. gen. all Ann's of m_b^n are fin. dim~~
- $\mathbb{C}[\mathfrak{h}^*/W]$ acts w. gen. e-value b.

} will see that this implies fin gen / $\mathbb{C}[\mathfrak{h}^*]$ – insert (*) here

$\bullet^*: M\to \mathrm{Hom}(M,\mathbb{C})$: $\mathcal{O}_{c^\vee}(\mathfrak{h}^*,W)_b\xrightarrow{\sim}\mathcal{O}_c(\mathfrak{h},W)^{\wedge_b}$] ← cor. of (*)

Observation: $*$ intertwines E_b & $\bullet^{\wedge_b}$: $(M^{\wedge_b})^* = E_b(M^*)$ $(M\in \mathcal{O}_c(\mathfrak{h},W))$.

$(M^{\wedge_{0,y}})^* = E_0(M^*)$, $M \in \mathcal{O}_{c^\vee}(\mathfrak{h}^*,W)_b$.

So $E_0(N)^* = (N^*)^{\wedge_{0,y}}$ $(N \in \mathcal{O}_c^{\wedge_b} \Rightarrow N^* \in \mathcal{O}_{c^\vee}(\mathfrak{h}^*,W)_b)$.

(*) ~~Problem Establish equivalence~~ $\cancel{\mathcal{O}_{c^\vee}(\mathfrak{h}^*,W)_b \xrightarrow{\sim} \mathcal{O}_{c^\vee}(\mathfrak{h}^*_{\underline{W}},\underline{W})}$ ~~using~~ $\cancel{\mathcal{O}}$

~~Cor:~~ $\cancel{M\in\mathcal{O}_{c^\vee}(\mathfrak{h}^*,W)_b}$ ~~has fin length~~;

• all simples are fin. gen / $\mathbb{C}[\mathfrak{h}^*]\#W$ (by spaces of "sing. vectors")

$\Rightarrow$ all objects in $\mathcal{O}_{c^\vee}(\mathfrak{h}^*,W)_b$ are fin. gen / $\mathbb{C}[\mathfrak{h}^*]$.

~~So~~: Functor $N \mapsto (N^*)^{\wedge_{0,y}}$ is exact $\Rightarrow$ $E_0(N)$ is exact.

3) Properties of Res & Ind

3.1) Behavior on K_0

Formal character: $M = \bigoplus_\alpha M_\alpha$ – gen. e-space for Euler element h

$\rightsquigarrow$ $\operatorname{ch} M = \sum_\alpha [M_\alpha]_W \cdot e^\alpha$

W-module

class in $K_0(W\text{-mod})$

Problem 2: • $\operatorname{ch} M_1 = \operatorname{ch} M_2 \Leftrightarrow [M_1] = [M_2]$ – classes in $K_0(\mathcal{O}_c)$

• $\operatorname{ch}\Delta(E) = \operatorname{ch}\nabla(E)$

Identify $K_0(\mathcal{O}_c)$ w. $K_0(W\text{-mod})$ via $[\Delta_c(E)] \longmapsto [E]_W$.

Prop (Bezrukavnikov–Etingof): $[\operatorname{Res}^W_{\underline{W}}] = [\operatorname{Res}^W_{\underline{W}}]_W$, $[\operatorname{Ind}^W_{\underline{W}}] = [\operatorname{Ind}^W_{\underline{W}}]_W$.

Proof: Res: $\Delta_c(E) \xmapsto[\wedge_b]{} \mathbb{C}[\mathfrak{h}]^{\wedge W b}\otimes E \xmapsto[e(\underline{W})]{} \mathbb{C}[\mathfrak{h}]^{\wedge_b}\otimes E \longmapsto$

$\mathbb{C}[\mathfrak{h}_{\underline{W}}]^{\wedge_0}\otimes E \longmapsto \mathbb{C}[\mathfrak{h}_{\underline{W}}]\otimes E$. – as $\mathbb{C}[\mathfrak{h}_{\underline{W}}]\#\underline{W}$-module

Problem 3: Let $M \in \mathcal{O}_c$ be such that $M \simeq \mathbb{C}[\mathfrak{h}]\otimes E$ $(E \in W\text{-mod})$, then $[M] = [\Delta_c(E)]$ and M is Δ-filtered

$\Rightarrow [\operatorname{Res}^W_{\underline{W}}] = [\operatorname{Res}^W_{\underline{W}}]_W$.

For Ind's: define $(\cdot,\cdot)$ on $K_0(\mathcal{O}_c)$ by $([M],[N]) = \sum_i (-1)^i \dim \operatorname{Ext}^i(M,N)$

– well-def.; $([\Delta(E)],[\Delta(E')]) = ([\Delta(E)],[\nabla(E')]) = \delta_{E,E'}$.

So $K_0(\mathcal{O}_c) \xrightarrow{\sim} K_0(W\text{-mod})$ preserves $(\cdot,\cdot)$; $\operatorname{Ind}^W_{\underline{W}}$ is exact & right adj. to $\operatorname{Res}^W_{\underline{W}}$ $\Rightarrow$ $[\operatorname{Ind}^W_{\underline{W}}]$ is adj-t to $[\operatorname{Res}^W_{\underline{W}}]$. Since $[\operatorname{Ind}^W_{\underline{W}}]_W$ is adj-t to $[\operatorname{Res}^W_{\underline{W}}]_W$, we are done by above $\square$

Rem: Functors $\mathrm{Ind}_{\underline{W}}^{W}$, $\mathrm{Res}_{\underline{W}}^{W}$ are defined using b: different choices of b (w. $W_b = \underline{W}$) give isomorphic functors ($\exists$ flat connection)

3.2) Ind/Res & KZ

Thm (Shan) $\mathrm{Ind}_{\underline{W}}^{W} \circ \underline{KZ} = KZ \circ \mathrm{Ind}_{\underline{W}}^{W}$, $\mathrm{Res}_{\underline{W}}^{W} \circ KZ = \underline{KZ} \circ \mathrm{Res}_{\underline{W}}^{W}$.

Will use: $\mathrm{End}(\mathrm{Ind}_{\underline{W}}^{W})$ same for $\mathcal{H}_c(\underline{W})$-mod & $\mathcal{O}_c$ - b/c

KZ is fully faithful on projectives

3.3) $\mathrm{Ind}_{\underline{W}}^{W}$ & $\mathrm{Res}_{\underline{W}}^{W}$ on $\mathcal{O}_c$ are biadjoint (cor. of 3.2 - Shan)

3.4) $\mathrm{Ind}_{\underline{W}}^{W}$ & $\mathrm{Res}_{\underline{W}}^{W}$ preserve Δ-filt. objects (I.L.)

Next time: categorical actions on $\mathcal{O}_c$ for $G(\ell,1,n)$

$\mathcal{O}_c = \bigoplus_{n=0}^{+\infty} \mathcal{O}_c(G(\ell,1,n))$

~~Recall $q \in \mathbb{C}^\times$ - q not root of 1 - action of $\mathfrak{sl}_\infty$~~

~~q-primit. e-th root of 1~~

Functors: direct summands of $\bigoplus_n \mathrm{Res}_n^{n-1} : \mathcal{O}_c(G(\ell,1,\bullet)) \to \mathcal{O}_c(G(\ell,1,\bullet-1))$

$\bigoplus_n \mathrm{Ind}_n^{n-1} : \mathcal{O}_c(G(\ell,1,\bullet)) \to \mathcal{O}_c(G(\ell,1,\bullet+1))$