SRA, Lec 23.
Categorical Kac-Moody actions.

1) Cyclotomic Hecke algebras
2) Categorical $\hat{\mathfrak{sl}}_e$-actions
3) Category $\mathcal{O}$.

1.1) Affine HA: Braid group of type $G(\zeta;n)$: $B^{aff}(n) = \langle T_0, \dots T_{n-1}\rangle$ mod rel-ns:
$T_iT_j = T_jT_i$, $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$, $i>0$, $T_0T_1T_0T_1 = T_1T_0T_1T_0$.
$q \in \mathbb{C}^\times \rightsquigarrow \mathcal{H}_q^{aff}(n) = \mathbb{C}B^{aff}(n)/((T_i-q)(T_i+1), i=1,\dots n-1)$.
Alt. presentation: $X_1 = T_0$, $X_2 = q^{-1}T_1X_1T_1$, $X_3 = q^{-1}T_2X_2T_2, \dots X_n = q^{-1}T_{n-1}X_{n-1}T_{n-1}$
Rel-ns: $X_iX_j = X_jX_i$ $(i=1, j=2$: $T_0T_1T_0T_1 = T_1T_0T_1T_0)$, $T_iX_iT_i = qX_{i+1}$
$T_iX_j = T_jX_i$ $(i-j\neq 0,1)$ + X_i is invertible
~~Fact~~ So $\mathcal{H}_q^{aff}(n) = \mathbb{C}\langle T_1,\dots T_{n-1}, X_1^{\pm1},\dots X_n^{\pm1}\rangle$/rel-ns for T_i's + rel-ns for X_i's & T_i's above

Important formula: $p\in\mathbb{C}[X_1^{\pm1},\dots X_n^{\pm1}] \rightsquigarrow s_i(p)(X_1,\dots X_n) = p(X_1,\dots X_{i+1}, X_i, \dots X_n)$

(1) $$T_ip - s_i(p)T_i = (q-1)\frac{p-s_i(p)}{1-X_iX_{i+1}^{-1}}$$

Cor: $\mathbb{C}[X_1^{\pm1},\dots X_n^{\pm1}]^{\mathfrak{S}_n}$ is central
Basis: ~~Prop~~ $w\in\mathfrak{S}_n$, $w = s_{i_1}\dots s_{i_k}$ - reduced expression (= min k)
$\rightsquigarrow T_w = T_{i_1}\dots T_{i_k}$ - well-defined
Prop: El-ts $X_1^{m_1}\dots X_n^{m_n}T_w$, $(m_1,\dots m_n\in\mathbb{Z}, w\in\mathfrak{S}_n)$ - basis in $\mathcal{H}_q^{aff}(n)$
(same true for $T_wX_1^{m_1}\dots X_n^{m_n}$)
Sketch of proof: i): prove that el-ts span $\mathcal{H}_q^{aff}(n)$
ii) $\mathcal{H}_q^{aff}(n) \curvearrowright \mathbb{C}[X_1^{\pm1},\dots X_n^{\pm1}]$ (using (1) w. $T_i\cdot 1 = q$)
el-ts act by lin. indep. operators $\square$
Cor: $\mathcal{H}_q^{aff}(n-1)\hookrightarrow\mathcal{H}_q^{aff}(n)$ $(T_i\mapsto T_i, X_i\mapsto X_i)$

1.2) Cyclotomic HA: $\underline{q} = (q_0,\dots q_{\ell-1})\in(\mathbb{C}^\times)^\ell \rightsquigarrow \mathcal{H}_q^{\underline{q}}(n) = \mathcal{H}_q^{aff}(n)/\prod_{i=0}^{\ell-1}(X_1-q_i)$
Thm (Ariki-Koike) $X_1^{m_1}\dots X_n^{m_n}T_w$ $(0\le m_i\le \ell-1, \forall i, w\in\mathfrak{S}_n)$
is basis in $\mathcal{H}_q^{\underline{q}}(n)$

Cor: $\mathcal{H}_q^{\underline{q}}(n-1) \hookrightarrow \mathcal{H}_q^{\underline{q}}(n)$

Trace: $\tau: H_q^{\underline{q}}(n) \longrightarrow \mathbb{C}$ $\tau(X_1^{m_1}\dots X_n^{m_n} T_w) = \delta_{m_1,0}\dots\delta_{m_n,0}\,\delta_{w,1}$.

Thm (Malle, Mathas) $(A,B) := \tau(AB)$ is symm. non-deg. form

Cor: $A \mapsto (A,\cdot)$ is iso $\mathcal{H}_q^{\underline{q}}(n) \underset{Bimod}{\xrightarrow{\sim}} \mathcal{H}_q^{\underline{q}}(n)^*$ ($\mathcal{H}_q^{\underline{q}}(n)$ is symmetr. alg.).

1.3) Res & its decomp-n. (due to Ariki)

From now on: q = primit. $\sqrt[e]{1}$, $q_i = q^{s_i}$, $s_i \in \mathbb{Z}$, $\mathcal{H}_q^{\underline{s}}(n) := \mathcal{H}_q^{\underline{q}}(n)$

$M \in \mathcal{H}_q^{\underline{s}}(n)$-mod: want study M induct-ly -restr. to $\mathcal{H}_q^{\underline{s}}(n-1)$

$[X_n, \mathcal{H}_q^{\underline{s}}(n-1)] = 0 \Rightarrow X_n \curvearrowright M$ by $\mathcal{H}_q^{\underline{s}}(n-1)$-lin. endom-s

(1) $\Longrightarrow$ e-vals $\in \{\sqrt[e]{1}\}$ (proved by ind. on n)

$i \in \mathbb{Z}/e\mathbb{Z} \rightsquigarrow E_i(M)$ = gen e-space for X_n w. e-val q^i

E_i -exact endof-r of $\mathcal{C} = \bigoplus_{n=0}^{+\infty} \mathcal{C}_n$, $\mathcal{C}_n = \mathcal{H}_q^{\underline{s}}(n)$-mod ($E_i|_{\mathcal{C}_0} = 0$).

$\bigoplus_{i\in\mathbb{Z}/e\mathbb{Z}} E_i = E \left(= \bigoplus_{n=0}^{+\infty} Res_n^{n-1}\right)$

Rem: Have $X \in End(E)$ -coming from X_n.

$T \in End(E^2)$, $E^2 = \bigoplus Res_n^{n-2}$, coming from T_{n-1}.

$X \rightsquigarrow 1_E X,\ X1_E \in End(E^2)$ ($1_E X$ acts by X_n, $X1_E$ by X_{n-1})

and sim. $1_E T, T1_E \in End(E^3)$ satisf.:

$(1_E X)(X1_E) = (X1_E)(1_E X)$ in $End(E^2)$

$T(X1_E)T = q(1_E X)$ in $End(E^2)$

$(T-q)(T+1) = 0$ — . — . —.

$(T1_E)(1_E T)(T1_E) = (1_E T)(T1_E)(1_E T)$ in $End(E^3)$

$\rightsquigarrow$ ~~[illegible]~~ $\mathcal{H}_q^{aff}(m) \longrightarrow End(E^m)$ -alg. homom.

1.4) Ind & its decomp

$F = \bigoplus_{n=0}^{+\infty} Ind_n^{n+1}$ - left & right adj. to E

$E = \bigoplus E_i \rightsquigarrow$ right adjunction $F = \bigoplus_{i\in\mathbb{Z}/e\mathbb{Z}} F_i$

Claim: F_i is also left adj-t of E_i

Proof: $Z = X_1 \dots X_n \in$ center of $\mathcal{H}_q^{aff}(n)$, $M \in \mathcal{H}_q^{\underline{q}}(n) \rightsquigarrow M_\alpha := \{m \in M \mid (Z-\alpha)^N m = 0, m \gg 0\} \rightsquigarrow M = \bigoplus_\alpha M_\alpha$ - decomp. into $\mathcal{H}_q^{\underline{q}}(n)$-mod $\rightsquigarrow \mathcal{C} = \bigoplus_\alpha \mathcal{C}_\alpha$

$\mathcal{C}_\alpha = \{M \mid M = M_\alpha\}$; ~~$E_i : M_\alpha \to M_{q^{-i}}$~~ $E_i(M_\alpha) = E(M)_{q^{-i}\alpha}$

$\Rightarrow E_i : \mathcal{C}_\alpha \to \mathcal{C}_{q^{-i}\alpha}$; right adjointness $F_i : \mathcal{C}_\alpha \to \mathcal{C}_{q^i\alpha}$, since $\bigoplus F_i$ is left adj to $\bigoplus E_i$, see that F_i is left adj. to E_i.

1.5) K_0.

$K_0(W\text{-mod})$: $G(\ell,1,n)$-irreps $\overset{\sim}{\longleftrightarrow}$ ℓ-multipartitions $\lambda = (\lambda^{(0)}, \dots \lambda^{(\ell-1)})$ of n

$\lambda^{(i)} \rightsquigarrow$ rep-n $S_{\lambda^{(i)}}$ of $\mathfrak{S}_{|\lambda^{(i)}|} \rightsquigarrow$ rep-n $S_{\lambda^{(i)}}(i)$ of $G(\ell,1,|\lambda^{(i)}|)$

w. $\mathfrak{S}_{|\lambda^{(i)}|}$ act. as before, $\eta \in$ copy of $\mathbb{Z}_\ell$ - by η^i.

$\rightsquigarrow S_\lambda = \operatorname{Ind}_{\prod G(\ell,1,|\lambda^{(i)}|)}^{G(\ell,1,n)} \boxtimes_{i=0}^{\ell-1} S_{\lambda^{(i)}}(i)$

Tits deform argument: for $q, \underline{q}$ generic $K_0(\mathcal{H}_q^{\underline{q}}(n)\text{-mod}) = K_0(G(\ell,1,n)\text{-mod})$

$\rightsquigarrow$ irrep. S_λ of $\mathcal{H}_q^{\underline{q}}(n)$-mod $\rightsquigarrow$ ~~cl~~ well-def. class $[S_\lambda]$ for any $q, \underline{q}$.

$[F][S_\lambda] = \bigoplus_\mu [S_\mu]$ - μ obt. from λ by adding a box.

$[E][S_\lambda] = \bigoplus_{\mu'} [S_{\mu'}]$ μ' - - - - - - - removing - - -

~~A box~~ $\alpha \in \mathbb{Z}/e\mathbb{Z}$: a box in jth diagram is an α-box if

$y - x + s_j \equiv \alpha$ mod e ($x = \#$ of row, $y = \#$ column)

first prove analog for $q, \underline{q}$ generic and then specialize

$[F_i][S_\lambda] = \bigoplus_\mu [S_\mu]$ - - - - - - - i-box

$[E_i][S_\lambda] = \bigoplus_\mu [S_\mu]$ - - - - - - - - -

Can define $\hat{\mathfrak{sl}}_e$-action on space w. basis of all $\lambda = (\lambda^{(0)}, \dots, \lambda^{(\ell-1)})$ by these formulas - get level ℓ Fock space w. multicharge $\underline{s}$. $K_0(\mathcal{H}_q^{\underline{s}}\text{-mod})$ is quotient of this - irred rep-n $\underbrace{\ }_{L(\Lambda)}$ w. highest weight $\Lambda = \sum_{i=0}^{\ell-1} \Lambda_{s_i}$ ← fund. weight

2) $\mathcal{C}$ - abelian, artinian cat w. enough projectives equipped w. functors E_i, F_i $(i \in \mathbb{Z}_e)$ and $X \in \operatorname{End}(E)$, $T \in \operatorname{End}(E^2)$ $(E = \bigoplus_{i \in \mathbb{Z}/e\mathbb{Z}} E_i)$

These data is categorical $\hat{\mathfrak{sl}}_e$-action if:

(1) E_i is biadj. to F_i

(2) E_i, F_i define $\hat{\mathfrak{sl}}_e$-action on $K_0(\mathcal{C})$, integrable

(3) class of $\overset{\text{any}}{\text{simple}}$ is a weight vector.

(4) X, T satisfy Hecke rel-ns (w. $q = e^{\sqrt{-1}}$) + E_i - e-functor $\underbrace{\text{w.}}_{\text{for } T}$ e-value q^i

Rem: (3) holds for $\mathcal{C} = \bigoplus_{n=0}^{+\infty} \mathcal{H}_q^{s}(n)$-mod (from action of center in $\mathcal{H}_q^{aff}(n)$)

Other examples: • category $\mathcal{O}$ for $\mathfrak{gl}_n$ and its ramifications

• Cherednik cat-y $\mathcal{O}$ (below)

3) Have $\mathcal{O}(n) = \mathcal{O}_c(\mathbb{C}^n, G(\ell, 1, n))$ w. c giving $q, \underline{q}$ as before

$\mathcal{O} = \bigoplus_{n=0}^{+\infty} \mathcal{O}(n)$ w. endofunctors ${}^{\mathcal{O}}E, {}^{\mathcal{O}}F$

Have $KZ: \mathcal{O} \to \mathcal{C}$ intertw. E, F + f. faith. on $\mathcal{O}$-proj. $\rightsquigarrow$

$End({}^{\mathcal{O}}E) = End(E)$, $End({}^{\mathcal{O}}F) = End(F) \rightsquigarrow X \in End(E)$

$+ \rightsquigarrow T \in End(E^2)$

$\rightsquigarrow E_i, F_i$

$[KZ(\Delta_c(\lambda))] = [S(\lambda)] \rightsquigarrow$ so action of $[E_i], [F_i]$ on $[\Delta_c(\lambda)]$ as before

So $K_0(\mathcal{O}) =$ Fock space and $[KZ] =$ projection from Fock space to $L(\Lambda)$