LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

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3. McKay correspondence upgraded

3.1. **Categorical quotients.** Let us start by reminding a few general definitions regarding algebraic groups. By an algebraic group one means a group that is also an algebraic variety in such a way that the group structure maps (the product and the inverse) are morphisms of algebraic varieties. For example, $\mathrm{GL}_n(\mathbb{C})$ is an algebraic group. Homomorphisms of algebraic groups as well as their actions are also supposed to be morphisms of algebraic varieties. Hence one has a notion of a representation of an algebraic group (a homomorphism $G \to \mathrm{GL}(V)$), such representations are usually called *rational*.

An algebraic group is called *reductive* if any its rational representation is completely reducible. Any finite group is reductive. In fact, any general linear group and any product of those are reductive as well, this was first proved by Weyl.

Let us return to representation of quivers. We have mentioned that a basic problem is to describe the orbits of $\mathrm{GL}(v)$ on $\mathrm{Rep}(Q, v)$, i.e., to describe a *quotient* of $\mathrm{Rep}(Q, v)$ by the $\mathrm{GL}(v)$-action. We can try to find an algebraic variety parameterizing the orbits. The field of Mathematics that tries to construct quotients for algebraic group actions is called (Geometric) Invariant theory. Basically, it offers two approaches, both work for reductive groups. A simpler one produces so called *categorical quotients*. We will consider it right below. Another one, *GIT quotients*, will be considered later.

Let X be an affine algebraic variety (in fact, one can take any affine scheme of finite type) and G be a reductive algebraic group acting on X. Let us try to produce a quotient as an affine algebraic variety, say Y. Then we should have a morphism $\varphi : X \to Y$ that is G-invariant, i.e., constant on the orbits. Consider the corresponding homomorphism $\varphi^* : \mathbb{C}[Y] \to \mathbb{C}[X]$. The condition that φ is G-invariant is equivalent to the condition that the pull-back of any function from $\mathbb{C}[Y]$ is G-invariant, i.e., $\varphi^*(\mathbb{C}[Y]) \subset \mathbb{C}[X]^G$. As in the case of finite groups, there is a general theorem again due to Hilbert saying that the algebra $\mathbb{C}[X]^G$ is finitely generated. So we can form an affine variety with algebra of functions equal to $\mathbb{C}[X]^G$. This variety is called the *categorical quotient* (for the G-action on X) and is denoted by $X//G$. The reason for the name is a universality property of the natural morphism $\pi : X \to X//G$ induced by the inclusion $\mathbb{C}[X]^G \subset \mathbb{C}[X]$ (and called the quotient morphism): for any other G-invariant morphism $\varphi : X \to Y$ of affine varieties there is a unique morphism $\psi : X//G \to Y$ such that $\varphi = \psi \circ \pi$. In other words, $X//G$ provides a finest parametrization of orbits that we can get using an affine variety (in fact, one can drop the condition of being affine).

However, sometimes such parametrization is pretty useless. For example, let $X = \mathbb{C}^n$ and let the one-dimensional torus $\mathbb{C}^\times := \mathrm{GL}_1(\mathbb{C}) = \{z \in \mathbb{C} | z \neq 0\}$ act on X by $t.(x_1, \ldots, x_n) = (tx_1, \ldots, tx_n)$.

Exercise 3.1. *Prove that there are no non-constant invariant polynomials for this action.*

So $X//G$ in our example is just a point.

Still the quotient morphism has some remarkable properties proved by Hilbert.

Theorem 3.1. *Let X be an affine algebraic variety, G be a reductive algebraic group acting on X and $\pi : X \to X//G$ denote the quotient morphism. Then the following is true.*

- *π is surjective.*
- *Every fiber of π contains a single closed orbit.*
- *Let $X_1 \subset X$ be a closed subvariety (or, more generally, a closed subscheme). Then $\pi(X_1) \subset X//G$ is closed and a natural morphism $X_1//G \to \pi(X_1)$ is an isomorphism.*

Exercise 3.2. *Use the theorem to show that the closure of any orbit contains a unique closed orbit.*

So the categorical quotient parameterizes the closed orbits for the action of G on X.

Let us provide an example. Consider the adjoint action of $G := \mathrm{GL}_n(\mathbb{C})$ on the space $X := \mathrm{Mat}_n(\mathbb{C})$ of $n \times n$-matrices by conjugations.

Problem 3.3. *Show that the algebra of invariants $\mathbb{C}[X]^G$ is generated by the coefficients of the characteristic polynomial of a matrix and is isomorphic to the algebra of polynomials in n variables. A hint: consider the restriction to the subspace of diagonal matrices.*

So any fiber of the quotient morphism consists of all matrices with a prescribed collection of eigenvalues.

Problem 3.4. *Show that every fiber indeed contains a single closed orbit and that this orbit consists of diagonalizable matrices.*

3.2. **Parametrization of representations.** An interesting special case is the construction of the representation varieties that parameterize (or try to parameterize) representations of a given associative algebra up to an isomorphism.

Take a finitely presented unital associative algebra $\mathcal{A}$. It can be presented as the quotient of a free algebra $\mathbb{C}\langle x_1, \ldots, x_n\rangle$ by some relations $f_1, \ldots, f_m$. Therefore representations of $\mathcal{A}$ in $\mathbb{C}^N$ are in one-to-one correspondence with the N-tuples $(M_1, \ldots, M_n)$ of matrices subject to $f_i(M_1, \ldots, M_N) = 0, i = 1, 2, \ldots, m$. So the representations are parameterized by the points of the subvariety in $\mathrm{Mat}_N(\mathbb{C})^n$ defined by the polynomials above. Denote this subvariety by $\mathrm{Rep}(\mathcal{A}, N)$.

Two representations $(M_1, \ldots, M_n), (M_1', \ldots, M_n')$ are isomorphic if and only if they lie in the same $G := \mathrm{GL}_N(\mathbb{C})$-orbit for the action given by $g.(M_1, \ldots, M_n) = (gM_1g^{-1}, \ldots, gM_ng^{-1})$. We can parameterize the closed orbits for this action by points of the categorical quotient $\mathrm{Rep}(\mathcal{A}, N)//G$. The question is to find some representation theoretic description of representations with closed orbits. It turns out that the description is pretty elegant.

Theorem 3.2. *The G-orbit of a representation is closed if and only if the representation is semisimple (=completely reducible).*

Exercise 3.5. *Show the "only if" part: if the orbit is closed, then the representation is semisimple.*

The proof of the other implication is harder, it is based on the following general result from Invariant theory called the *Hilbert-Mumford* criterium (proved in the full generality by Mumford; a relatively elementary proof can be found in [B]).

Theorem 3.3. *Let a reductive group G act on an affine variety X. Let $x, y \in X$ be such that Gy is the closed orbit in $\overline{Gx}$. Then there is a one-parameter subgroup (=algebraic group homomorphism) $\gamma : \mathbb{C}^\times \to G$ such that $\lim_{t\to 0} \gamma(t).x \in Gy$.*

Problem 3.6. *Deduce the if part of Theorem 3.2 from the Hilbert-Mumford criterium.*

3.3. **McKay correspondence upgraded: an overview.** Let $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ be a finite subgroup. To this subgroup we can assign an un-oriented graph $\underline{Q}$ with vertices $0, \ldots, r$, where $N_0, \ldots, N_r$ are the irreducible representations of Γ with N_0 being the trivial representation. The number of edges between i and j is $m_{ij} := \dim \mathrm{Hom}_\Gamma(\mathbb{C}^2 \otimes N_i, N_j)$. This is of course not yet a quiver, as the latter has to be oriented. For what follows it will be convenient to *choose* an orientation on $\underline{Q}$ making it into a quiver. Then we consider the double quiver: for each arrow a in $\underline{Q}_1$ we add a new arrow a^* with opposite orientation. So the number of arrows from i to j is m_{ij}. Also recall the dimension vector $\delta = (\delta_i)_{i \in Q_0}$ with $\delta_i = \dim N_i$.

A variety that we are going to produce from Q is as follows. We consider the representation space

$$\mathrm{Rep}(Q, \delta) \cong \bigoplus_{a \in Q_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\delta_{t(a)}}, \mathbb{C}^{\delta_{h(a)}}) = \bigoplus_{i,j=0}^{r} \mathrm{Hom}(\mathbb{C}^{\delta_i}, \mathbb{C}^{\delta_j})^{m_{ij}}$$

that come equipped with an action of $\mathrm{GL}(\delta)$ given by $(g_i)_{i \in Q_0}(x_a)_{a \in Q_1} = (g_{h(a)} x_a g_{t(a)}^{-1})$. Let $\mathfrak{gl}(\delta)$ denote the Lie algebra of $\mathrm{GL}(\delta)$, i.e., $\mathfrak{gl}(\delta) := \prod_{i \in Q_0} \mathfrak{gl}_{\delta_i}$, where $\mathfrak{gl}_{\delta_i}$ is the Lie algebra of all matrices. We have a quadratic map $\mu : \mathrm{Rep}(Q, \delta) \to \mathfrak{gl}(\delta)$, $\mu = (\mu_i)_{i \in Q_0}$, where

$$\mu_i\left((x_a)_{a \in Q_1}\right) = \sum_{a \in \underline{Q}_1, t(a) = i} x_{a^*} x_a - \sum_{a \in \underline{Q}_1, h(a) = i} x_a x_{a^*}.$$

For example, consider Γ of type A_r. Then we can orient $\underline{Q}$ counterclockwise, let a_i denote the arrow from i to $i-1$ and let a_i^* be the opposite arrow. Then $\mu_i = x_{a_i^*} x_{a_i} - x_{a_{i+1}} x^*_{a_{i+1}}$.

The variety we are interested in is $\mu^{-1}(0)//G$. The following is what we mean by the upgraded McKay correspondence.

Theorem 3.4. *We have an isomorphism of varieties* $\mathbb{C}^2/\Gamma \cong \mu^{-1}(0)// \mathrm{GL}(\delta)$.

The proof of the theorem consists of several steps. To describe them we need some more notation. Let A be an algebra acted on by Γ. Recall the smash-product $A\#\Gamma$ which contains $\mathbb{C}\Gamma$ as a subalgebra. We have a distinguished representation of that subalgebra in itself (by left multiplications). We consider the variety $\mathrm{Rep}_\Gamma(A\#\Gamma, \mathbb{C}\Gamma)$ of all representations of $A\#\Gamma$ that restrict to $\mathbb{C}\Gamma$ as that fixed representation. Similarly to the previous section, this set carries a natural structure of an algebraic variety. Since the representations are fixed on $\mathbb{C}\Gamma$, they are isomorphic if and only if they lie in the same orbit under a natural action of the group $\mathrm{GL}(\mathbb{C}\Gamma)^\Gamma$ of all Γ-equivariant elements of $\mathrm{GL}(\mathbb{C}\Gamma)$.

Here are the steps we use to prove Theorem 3.4:

Step 1: Identify the isomorphism classes of semisimple representations of $\mathrm{Rep}_\Gamma(\mathbb{C}[x,y]\#\Gamma, \mathbb{C}\Gamma)$ with $\mathbb{C}^2/\Gamma$ (via taking "central character").

Step 2: Identify $\mathrm{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma, \mathbb{C}\Gamma)$ with $\mathrm{Rep}(Q,\delta)$ so that the action of $\mathrm{GL}(\mathbb{C}\Gamma)^\Gamma$ on the former becomes the action of $G = \mathrm{GL}(\delta)$ on the latter. In fact, such an identification is not unique so we will need to choose a nice one.

Step 3: $\mathrm{Rep}_\Gamma(\mathbb{C}[x,y]\#\Gamma, \mathbb{C}\Gamma)$ is a subvariety (more precisely, a subscheme) in $\mathrm{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma, \mathbb{C}\Gamma)$ given by the condition that $[x,y] = xy - yx$ acts by 0. To prove Theorem 3.4 it remains to show that, under a suitable identification of $\mathrm{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma, \mathbb{C}\Gamma)$ with $\mathrm{Rep}(Q,\delta)$, the assignment that maps a representation φ to $\varphi(xy - yx)$ becomes μ (this includes the claim that these can be viewed as maps to the same space). To prove the coincidence is the most delicate and technical step.

Then Theorem 3.4 will just follow from the fact that the categorical quotient parameterizes the semisimple representations (well, not quite, we still need to establish a few things: that $\mu^{-1}(0)//G$ is a reduced scheme, that the bijection $\mathbb{C}^2/\Gamma \cong \mu^{-1}(0)//G$ that we are going to get is a morphism of varieties). We will make some remarks on these issues below.

3.4. **Step 1.** Recall that $\mathbb{C}[x,y]^\Gamma$ is embedded (and actually coincides with) the center of $\mathbb{C}[x,y]\#\Gamma$. According to the Schur lemma, any central element, say a, has to act by a scalar, say a_V, on an irreducible representation V. The map $a \mapsto a_V : \mathbb{C}[x,y]^\Gamma \to \mathbb{C}$ is easily seen to be an algebra homomorphism. So it defines a point, say p_V, in $\mathbb{C}^2/\Gamma$.

Lemma 3.5. *For each nonzero $p \in \mathbb{C}^2/\Gamma$ there is a unique (up to an isomorphism) irreducible representation V of $\mathbb{C}[x,y]\#\Gamma$ with $p_V = p$ and, moreover, V is isomorphic to $\mathbb{C}\Gamma$ as a representation of Γ.*

Proof. Consider the subalgebra $\mathbb{C}[x,y] \subset \mathbb{C}[x,y]\#\Gamma$. This algebra is commutative and so any its irreducible representation is 1-dimensional. So there is a point $\tilde{p} \in \mathbb{C}^2$ and a vector $v \in V$ such that $a.v = \tilde{p}(a)v$ for any $a \in \mathbb{C}[x,y]$ (in other words, v is an eigenvector for $\mathbb{C}[x,y]$ with "eigenvalue" $\tilde{p}$). Since $\mathbb{C}[x,y]^\Gamma$ (viewed as the center of $\mathbb{C}[x,y]\#\Gamma$) is contained in $\mathbb{C}[x,y]$ we see that $\tilde{p}|_{\mathbb{C}[x,y]^\Gamma} = p$ or, equivalently, $\tilde{p}$ lies in the Γ-orbit corresponding to p.

Next we remark that the subalgebra $\mathbb{C}[x,y] \subset \mathbb{C}[x,y]\#\Gamma$ is normalized by the adjoint action of Γ. It follows that the vector $gv, g \in \Gamma$, is again an eigenvector for $\mathbb{C}[x,y]$, whose eigenvalue is the point $g\tilde{p}$. The following exercise shows that all points $g\tilde{p}$ are distinct.

Exercise 3.7. *Show that the stabilizer in Γ of any nonzero point in $\mathbb{C}^2$ is trivial.*

The span of the vectors $g\tilde{p}, g \in \Gamma$, is clearly Γ-stable and also $\mathbb{C}[x,y]$-stable. So it is a submodule hence needs to coincide with V. Also the vectors $g\tilde{p}$ are linearly independent because their eigenvalues are pairwise different. This completes the proof. □

Now let us consider the irreducible representations V with $p_V = 0$.

Lemma 3.6. *Suppose that V is an irreducible representation of $\mathbb{C}[x,y]\#\Gamma$ such that $p_V = 0$. Then x, y act on V by 0 and the restriction of V on $\mathbb{C}\Gamma$ is irreducible.*

Proof. Similarly to the first paragraph of the proof of Lemma 3.5, all eigenvalues of $\mathbb{C}[x,y]$ on V are zero. So the common kernel V_0 of x, y in V is nonzero. But V_0 is Γ-stable and hence a $\mathbb{C}[x,y]\#\Gamma$-submodule. So $V = V_0$ and V_0 is an irreducible representation of Γ. □

Lemmas 3.5,3.6 allow to describe the semisimple representations in $\mathrm{Rep}_\Gamma(\mathbb{C}[x,y]\#\Gamma, \mathbb{C}\Gamma)$ (up to an isomorphism) completely. Such a representation is either an irreducible representation V with $p_V = 0$ or is isomorphic to $\mathbb{C}\Gamma$ with zero action of x and y. It follows that there is a bijection of the isomorphism classes of semisimple representations in $\mathrm{Rep}_\Gamma(\mathbb{C}[x,y]\#\Gamma, \mathbb{C}\Gamma)$ and $\mathbb{C}^2/\Gamma$.

3.5. **Step 2.** Let us discuss an alternative way to look at the representations from $\mathrm{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma, \mathbb{C}\Gamma)$. On any such representation, a Γ-action is already given and so we only need to specify the actions of x, y. To give such an action is the same as to give a linear map $\mathbb{C}^2 \otimes \mathbb{C}\Gamma \to \mathbb{C}\Gamma$ (where we view x, y as a basis in $\mathbb{C}^2$).

Exercise 3.8. *Such map extends to an action of $\mathbb{C}\langle x,y\rangle\#\Gamma\mathbb{C}\langle x,y\rangle\#\Gamma$ if and only if it is Γ-equivariant.*

So the representations we are interested in are parameterized by the points in the vector space $\mathrm{Hom}_\Gamma(\mathbb{C}^2\otimes\mathbb{C}\Gamma,\mathbb{C}\Gamma)$. The Γ-module $\mathbb{C}\Gamma$ is decomposed as $\bigoplus_{i=0}^r N_i\otimes N_i^*$, where Γ acts on the left factors and so, choosing bases in the spaces $N_i^*, i=0,\ldots,r$, we identify $\mathbb{C}\Gamma$ with $\bigoplus_{i=0}^r N_i^{\oplus\delta_i}$. We set $M_{ij}:=\mathrm{Hom}_\Gamma(\mathbb{C}^2\otimes N_i,N_j)$ so that $m_{ij}:=\dim M_{ij}$.

Exercise 3.9. *Show that*

$$\mathrm{Hom}_\Gamma(\mathbb{C}^2\otimes\mathbb{C}\Gamma,\mathbb{C}\Gamma)=\bigoplus_{i,j=0}^r M_{ij}\otimes\mathrm{Hom}_\mathbb{C}(N_i^*,N_j^*)=\bigoplus_{i,j=0}^r \mathrm{Hom}_\mathbb{C}(\mathbb{C}^{\delta_i},\mathbb{C}^{\delta_j})^{m_{ij}}.$$

We remark that the first equality here is a completely canonical isomorphism, while the second one is not: it requires choosing bases in, first, the spaces $N_i^*, i=0,\ldots,r$, and, second, the spaces M_{ij}. We recall that, with one exception, the latter are always one or zero dimensional and so a basis vector is defined uniquely up to proportionality. The exception is $\Gamma=\mathbb{Z}/2\mathbb{Z}$, here M_{12} is two dimensional. We will ignore this exception.

The previous exercise implies that the space of representations of $\mathbb{C}\langle x,y\rangle\#\Gamma$ is nothing else but $\mathrm{Rep}(Q,\delta)$.

If we decompose $\mathbb{C}\Gamma$ as $\bigoplus_{i=0}^r N_i^{\delta_i}$, then (using the Schur lemma) we see that $\mathrm{GL}(\mathbb{C}\Gamma)^\Gamma=\mathrm{GL}(\delta):=\prod_{i=0}^r\mathrm{GL}(\delta_i)$ and, under our identification of $\mathrm{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma,\mathbb{C}\Gamma)$ with $\mathrm{Rep}(Q,\delta)$, the $\mathrm{GL}(\mathbb{C}\Gamma)^\Gamma$-action on the former becomes the $\mathrm{GL}(\delta)$-action on the latter.

So our conclusion is that the semisimple representations in $\mathrm{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma,\mathbb{C}\Gamma)$ are parameterized up to an isomorphism by the points of $\mathrm{Rep}(Q,\delta)/\!/\,\mathrm{GL}(\delta)$.

References

[B] D. Birkes. *Orbits of linear algebraic groups.* Ann. Math. 93(1971), 459-475.