LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

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4. Deformed preprojective algebras

Recall that we have introduced the double McKay quiver Q, its representation space $\operatorname{Rep}(Q,\delta)$ acted on by the group $\operatorname{GL}(\delta)$ and also a somewhat mysterious quadratic map $\mu:\operatorname{Rep}(Q,\delta)\rightarrow\mathfrak{gl}(\delta)$. We have claimed that $\mathbb{C}^2/\Gamma\cong\mu^{-1}(0)/\!/\operatorname{GL}(\delta)$.

For this we have realized $\mathbb{C}^2/\Gamma$ as a "moduli space" for certain representations of $\mathbb{C}[x,y]\#\Gamma$. Namely, we considered the variety $\operatorname{Rep}_\Gamma(\mathbb{C}[x,y]\#\Gamma,\mathbb{C}\Gamma)$ of all representations of $\mathbb{C}[x,y]\#\Gamma$ whose restriction to $\mathbb{C}\Gamma$ is the representation by left multiplications. A few remarks about this choice are in order.

First, it is not necessary to fix an isomorphism of a representation of $\mathbb{C}[x,y]\#\Gamma$ with $\mathbb{C}\Gamma$, it is enough to consider all representations of $\mathbb{C}[x,y]\#\Gamma$ whose restriction to $\mathbb{C}\Gamma$ is isomorphic to the regular representation. In this case the symmetry group becomes bigger, $\operatorname{GL}(\mathbb{C}\Gamma)$, and is no longer identified with $\operatorname{GL}(\delta)$, but the "moduli space" (=the space parameterizing the representations) remains the same.

Second, let us explain why we choose to consider the representations in $\mathbb{C}\Gamma$ and not in some other Γ-module V. The reason is that if $\mathbb{C}\Gamma\not\hookrightarrow_\Gamma V$, then x,y act by 0 act on all simple $\mathbb{C}[x,y]\#\Gamma$-modules entering V (this follows from the classification of the irreducible $\mathbb{C}[x,y]\#\Gamma$-modules performed last time) and so we do not get an interesting moduli space.

To prove an isomorphism $\mathbb{C}^2/\Gamma\cong\mu^{-1}(0)/\!/\operatorname{GL}(\delta)$, it remains to do two steps: to show that $\operatorname{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma,\mathbb{C}\Gamma)$ with $\operatorname{Rep}(Q,\delta)$ and to show that the condition $\varphi(xy-yx)=0$ on $\varphi\in\operatorname{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma,\mathbb{C}\Gamma)$ is equivalent to $\mu(\varphi)=0$ (we do not prove the promised equality of $\varphi\mapsto\varphi(xy-yx)$ and μ, it is a subtle question, in what sense this equality holds).

4.1. Step 2.

We need an alternative way to look at the representations from $\operatorname{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma,\mathbb{C}\Gamma)$. On any such representation, a Γ-action is already given and so we only need to specify the actions of x,y. To give such an action is the same as to give a linear map $\mathbb{C}^2\otimes\mathbb{C}\Gamma\rightarrow\mathbb{C}\Gamma$ (where we view x,y as a basis in $\mathbb{C}^2$).

Exercise 4.1. *A map $\mathbb{C}^2\otimes\mathbb{C}\Gamma\rightarrow\mathbb{C}\Gamma$ extends to an action of $\mathbb{C}\langle x,y\rangle\#\Gamma$ if and only if it is Γ-equivariant.*

So the representations we are interested in are parameterized by the points in the vector space $\operatorname{Hom}_\Gamma(\mathbb{C}^2\otimes\mathbb{C}\Gamma,\mathbb{C}\Gamma)$. The Γ-module $\mathbb{C}\Gamma$ is decomposed as $\bigoplus_{i=0}^r N_i\otimes N_i^*$, where Γ acts on the left factors and so, choosing bases in the spaces $N_i^*, i=0,\ldots,r$, we identify $\mathbb{C}\Gamma$ with $\bigoplus_{i=0}^r N_i^{\oplus\delta_i}$. We set $M_{ij}:=\operatorname{Hom}_\Gamma(\mathbb{C}^2\otimes N_i,N_j)$ so that $m_{ij}:=\dim M_{ij}$.

Exercise 4.2. *Show that*

$$\begin{aligned}\mathrm{Hom}_\Gamma(\mathbb{C}^2\otimes\mathbb{C}\Gamma,\mathbb{C}\Gamma)&=\bigoplus_{i,j=0}^r M_{ij}\otimes \mathrm{Hom}_{\mathbb{C}}(N_i^*,N_j^*)\\ &=\bigoplus_{i,j=0}^r \mathrm{Hom}_{\mathbb{C}}(N_i^*,N_j^*)^{\oplus m_{ij}}=\bigoplus_{i,j=0}^r \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\delta_i},\mathbb{C}^{\delta_j})^{m_{ij}}.\end{aligned}$$

Note that the first equality is canonical, the second depends on the choice of a basis in M_{ij}, while the third depends on the choice of bases in N_i^*.

We recall that, with one exception, the space M_{ij} are always one or zero dimensional and so a basis vector is defined uniquely up to proportionality. The exception is $\Gamma=\mathbb{Z}/2\mathbb{Z}$, here M_{12} is two dimensional. We will ignore this exception.

The previous exercise implies that the space of representations of $\mathbb{C}\langle x,y\rangle\#\Gamma$ is nothing else but $\mathrm{Rep}(Q,\delta)$.

If we decompose $\mathbb{C}\Gamma$ as $\bigoplus_{i=0}^r N_i^{\delta_i}$, then (using the Schur lemma) we see that $\mathrm{GL}(\mathbb{C}\Gamma)^\Gamma=\prod_{i=0}^r\mathrm{GL}(N_i^*)=\mathrm{GL}(\delta):=\prod_{i=0}^r\mathrm{GL}(\delta_i)$ (where the first equality is canonical, while the second is not) and, under our identification of $\mathrm{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma,\mathbb{C}\Gamma)$ with $\mathrm{Rep}(Q,\delta)$, the $\mathrm{GL}(\mathbb{C}\Gamma)^\Gamma$-action on the former becomes the $\mathrm{GL}(\delta)$-action on the latter.

So our conclusion is that the semisimple representations in $\mathrm{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma,\mathbb{C}\Gamma)$ are parameterized up to an isomorphism by the points of $\mathrm{Rep}(Q,\delta)/\!/\mathrm{GL}(\delta)$.

That's what we need on Step 2. But actually, in order to accomplish Step 3 of our original program, we will need some ramifications of Step 2.

4.2. **Path algebras.** As in the case of a group or a Lie algebra, a representation of a quiver Q is the same as a module over a certain algebra. This algebra is called the *path algebra* of Q, is denoted by $\mathbb{C}Q$ and is constructed as follows. For a basis in $\mathbb{C}Q$ we will take all paths in Q, i.e., all sequences of arrows $p=(a_1,\ldots,a_k)$ such that $h(a_1)=t(a_2),\ldots,h(a_{k-1})=t(a_k)$. We set $t(p):=t(a_1),h(p)=h(a_k)$ and we say that p has length k. We also include empty paths $\epsilon_i,i\in Q_0$, with $t(\epsilon_i)=h(\epsilon_i):=i$. By definition, the product p_1p_2 of two pathes p_1,p_2 is zero if $h(p_2)\neq t(p_1)$ and is the concatenation of p_1 and p_2 else. For example, the path algebra of the Jordan quiver is the polynomial algebra in one variable. Another example: take the Dynkin quiver of type A_2, i.e, the quiver with two vertices, 1 and 2, and a single arrow a with $t(a)=1,h(a)=2$. The path algebra is three dimensional, its basis is ϵ_1,ϵ_2,a and the only nonzero products are $\epsilon_1^2=\epsilon_1,\epsilon_2^2=\epsilon_2,a\epsilon_1=a,\epsilon_2a=a$.

Exercise 4.3. *Show that $\mathbb{C}Q$ is associative and $\sum_{i\in Q_0}\epsilon_i$ is a unit in $\mathbb{C}Q$. Further, show that, as a unital associative algebra, $\mathbb{C}Q$ is generated by $\epsilon_i,i\in Q_0$, and $a\in Q_1$ subject to the relations $\epsilon_i\epsilon_j=\delta_{ij}\epsilon_i,\sum_{i\in Q_0}\epsilon_i=1,\epsilon_ia=\delta_{ih(a)}a,a\epsilon_i=\delta_{it(a)}a$.*

If (V_i,x_a) is a representation of Q, then the space $V=\bigoplus_i V_i$ is equipped with a unique $\mathbb{C}Q$-module structure such that ϵ_i acts by the projection to the summand V_i and a acts by x_a. Conversely, to a $\mathbb{C}Q$-module U one assigns a representation of Q with $V_i=\epsilon_iU$.

Finally, let us remark that the algebra $\mathbb{C}Q$ is graded, $\mathbb{C}Q=\bigoplus_{i=0}^{+\infty}(\mathbb{C}Q)^i$, with $(\mathbb{C}Q)^i$ being the linear span of all pathes with length i.

4.3. $\mathbb{C}Q$ **vs** $\mathbb{C}\langle x,y\rangle\#\Gamma$**.** Now we are going to relate $\mathbb{C}Q$ for the (doubled) McKay quiver Q to $\mathbb{C}\langle x,y\rangle\#\Gamma$. For this we will realize both as tensor algebras.

Namely, if we have an associative algebra A and its *bimodule* M, we can form the tensor products $M^{\otimes n} := M \otimes_A M \otimes_A \ldots \otimes_A M$ and hence also the tensor algebra $T_A(M) = \bigoplus_n M^{\otimes n}$. This algebra has a usual universal property.

To represent $\mathbb{C}\langle x, y\rangle \# \Gamma$ in this form we will take $A = \mathbb{C}\Gamma$ and $M := \mathbb{C}^2 \otimes \mathbb{C}\Gamma$, where the left A-action comes from the Γ-module structure on $\mathbb{C}^2 \otimes \mathbb{C}\Gamma$, while the right action is by right multiplications on the second factor.

To represent $\mathbb{C}Q$ in this form we take $A = (\mathbb{C}Q)^0 \cong \mathbb{C}^{Q_0}$ and $M = (\mathbb{C}Q)^1$, the span of all arrows with the bimodule structure coming from Exercise 4.3.

Exercise 4.4. *Use the universal properties of all algebras involved to show that* $\mathbb{C}\langle x, y\rangle \# \Gamma \cong T_{\mathbb{C}\Gamma}(\mathbb{C}^2 \otimes \mathbb{C}\Gamma)$ *and* $\mathbb{C}Q \cong T_{(\mathbb{C}Q)^0}(\mathbb{C}Q)^1$.

A relation between the algebras $\mathbb{C}\langle x, y\rangle \# \Gamma$ and $\mathbb{C}Q$ is as follows: there is an idempotent $f \in \mathbb{C}\Gamma$ such that $\mathbb{C}Q = f\mathbb{C}\langle x, y\rangle \# \Gamma f$. To prove this we will need to examine an interplay between the constructions of spherical subalgebras and of tensor algebras.

Now suppose that A is an arbitrary associative algebra and that $e \in A$ is an idempotent. Then we can form the spherical subalgebra eAe. The space Ae has commuting actions of A on the left and eAe on the right. We have a functor $\pi : A\text{-Mod} \to eAe\text{-Mod}$ that sends M to eM. On the other hand, consider a functor $\pi^!$ in the opposite direction given by $\pi^!(N) = Ae \otimes_{eAe} N$.

Exercise 4.5.

- *Show that π is an exact functor, that π can be written as $M \mapsto eA \otimes_A M$, and that $\pi^!$ is left adjoint to π.*
- *Suppose that $AeA = A$. Check the that if $\pi(M) = 0$, then $M = 0$. Further check that the natural homomorphism $Ae \otimes_{eAe} eM \to M$ is surjective. Finally, show that $Ae \otimes_{eAe} eM \to M$ is injective by applying π.*
- *Deduce that $Ae \otimes_{eAe} eA = A$ as bimodules.*

The previous exercise shows that the categories of modules for A and for eAe are equivalent. In this case one says that the algebras A and eAe are Morita equivalent (or that e induces a Morita equivalence).

Our goal now is to find an idempotent $f \in \mathbb{C}\langle x, y\rangle \# \Gamma$ such that $\mathbb{C}Q \cong f\mathbb{C}\langle x, y\rangle \# \Gamma f$. Recall that $\mathbb{C}\Gamma = \bigoplus_{i \in Q_0} \operatorname{End}(N_i^*)$. Pick a primitive idempotent (a.k.a. diagonal matrix unit) $f_i \in \operatorname{End}(N_i)$ and set $f := \sum_{i \in Q_0} f_i$. Then, obviously, $\mathbb{C}\Gamma f \mathbb{C}\Gamma = \mathbb{C}\Gamma$ and $f\mathbb{C}\Gamma f = \mathbb{C}^{Q_0}$. Let us compute $f(\mathbb{C}^2 \otimes \mathbb{C}\Gamma)f$. For this we will need to understand the structure of the bimodule $\mathbb{C}^2 \otimes \mathbb{C}\Gamma$ over the algebra $\mathbb{C}\Gamma = \bigoplus_{i \in Q_0} \operatorname{End}(N_i^*)$. We have $\mathbb{C}^2 \otimes \mathbb{C}\Gamma = \bigoplus_{i \in Q_0} \mathbb{C}^2 \otimes N_i \otimes N_i^*$, where the left action of Γ is on the first two factors, while the right action is on the third factor. Further, $\mathbb{C}^2 \otimes N_i = \bigoplus_j M_{ij} \otimes N_j$, where Γ acts trivially on the first factor. So $\mathbb{C}^2 \otimes \mathbb{C}\Gamma = \bigoplus_{ij} M_{ij} \otimes N_j \otimes N_i^* = \bigoplus_{ij} M_{ij} \operatorname{Hom}(N_j^*, N_i^*)$. The space $\operatorname{Hom}(N_j^*, N_i^*)$ has a natural left action of $\operatorname{End}(N_i^*)$ and a natural right action of $\operatorname{End}(N_j^*)$ that gives the structure of a $\bigoplus_{i \in Q_0} \operatorname{End}(N_i^*)$-bimodule on $\mathbb{C}^2 \otimes \mathbb{C}\Gamma$. So $f_{j'} \operatorname{Hom}(N_i^*, N_j^*) f_{i'} = \delta_{ii'}\delta_{jj'}\mathbb{C}$ and we get $f(\mathbb{C}^2 \otimes \mathbb{C}\Gamma)f = \bigoplus_{ij} M_{ij} = (\mathbb{C}Q)^1$.

The following exercise implies that $f(\mathbb{C}\langle x, y\rangle \# \Gamma)f = \mathbb{C}Q$.

Exercise 4.6. *Suppose e is an idempotent in A such that $AeA = A$. Show that the functor $M \mapsto eMe$ is an equivalence between the categories of A and eAe-bimodules intertwining the tensor products (meaning that $e(M \otimes_A N)e = eMe \otimes_{eAe} eNe$). Deduce that $eT_A(M)e$ is naturally identified $T_{eAe}(eMe)$.*

Now we can revisit the identification $\mathrm{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma,\mathbb{C}\Gamma)\cong \mathrm{Rep}(Q,\delta)$. The latter is the representation space of $\mathbb{C}Q$ in $\bigoplus_{i\in Q_0} N_i^*$. The isomorphism is induced by the map $M\mapsto fM$, where $f\in \mathrm{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma,\mathbb{C}\Gamma)$.

4.4. **Deformed preprojective algebras.** The identification of $\mathrm{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma,\mathbb{C}\Gamma)$ with $\mathrm{Rep}(Q,\delta)$ (as well as the isomorphism $f(\mathbb{C}\langle x,y\rangle\#\Gamma)f\cong \mathbb{C}Q$) depended on the choice of bases in the spaces N_i^* and also on the choice of bases in the spaces M_{ij}. We are going to prove that the condition that $\varphi(xy-yx)=0$ is equivalent to $\mu(\varphi)=0$ (under a suitable choice of a basis in M_{ij}).

In fact, we will prove a stronger result. Namely, recall the algebra $H_c=\mathbb{C}\langle x,y\rangle\#\Gamma/(xy-yx-c)_{\mathbb{C}\langle x,y\rangle\#\Gamma}$, where $c\in(\mathbb{C}\Gamma)^\Gamma$. Here and below the superscript $\mathbb{C}\langle x,y\rangle\#\Gamma$ after the brackets means that we take the two-sided ideal in $\mathbb{C}\langle x,y\rangle\#\Gamma$ generated by the element(s) in brackets. The algebra fH_cf is a quotient of $f(\mathbb{C}\langle x,y\rangle\#\Gamma)f=\mathbb{C}Q$ by the ideal $f(xy-yx-c)_{\mathbb{C}\langle x,y\rangle\#\Gamma}f$ and the question is how to describe the ideal explicitly. The answer is due to Crawley-Boevey and Holland and is as follows.

For $\lambda=(\lambda_i)_{i\in Q_0}$, define the *deformed preprojective algebra* Π^λ by as the quotient of $\mathbb{C}Q$ by the relations

$$\sum_{a\in\underline{Q}_1,t(a)=i} a^*a-\sum_{a\in\underline{Q}_1,h(a)=i} aa^*-\lambda_i\epsilon_i=0, \tag{1}$$

one for each $i\in Q_0$. Below we will write $[a^*,a]_i$ for $\sum_{a\in\underline{Q}_1,t(a)=i}a^*a-\sum_{a\in\underline{Q}_1,h(a)=i}aa^*$.

Theorem 4.1. *With a suitable choice with of bases in M_{ij}, the ideal $f(xy-yx-c)f$ is generated by the $[a^*,a]_i-\lambda_i\epsilon_i$, where $\lambda_i:=\mathrm{tr}_{N_i}(c)$.*

In particular, $xy-yx$ acts trivially on $M\in\mathrm{Rep}_\Gamma(\mathbb{C}\langle x,y\rangle\#\Gamma,\mathbb{C}\Gamma)$ if and only if M is annihilated by the ideal $(xy-yx)_{\mathbb{C}\langle x,y\rangle\#\Gamma}$ if and only if $fM\in\mathrm{Rep}(Q,\delta)$ is annihilated by $f(xy-yx)_{\mathbb{C}\langle x,y\rangle\#\Gamma}f$, i.e. by all elements $[a^*,a]_i\in\mathbb{C}Q$. By that element just acts by the operator $\mu_i\in\mathrm{End}(\mathbb{C}^{\delta_i})$.

4.5. **CBH lemma.** To prove Theorem 4.1 we will need a lemma from [CBH]. First, we need a concrete form of isomorphisms between $\mathrm{Hom}_\Gamma(\mathbb{C}^2\otimes N_i,N_j)$ and $\mathrm{Hom}_\Gamma(N_i,\mathbb{C}^2\otimes N_j)$. We view x,y as a basis in $\mathbb{C}^2$, identifying $\mathbb{C}^2$ with its dual via the symplectic form ω given by $\omega(y,x)=1=-\omega(x,y)$. Further, let $\zeta:=x\otimes y-y\otimes x\in\mathbb{C}^2\otimes\mathbb{C}^2$. We can view ω as a map $\mathbb{C}^2\otimes\mathbb{C}^2\to\mathbb{C}$, and ζ as a map $\mathbb{C}\to\mathbb{C}^2\otimes\mathbb{C}^2$. Both maps are Γ-equivariant if we view $\mathbb{C}$ as the trivial module.

Let M,M' be Γ-modules. Take $\psi\in\mathrm{Hom}_\Gamma(M,\mathbb{C}^2\otimes M')$. It gives rise to a map $1_{\mathbb{C}^2}\otimes\psi:\mathbb{C}^2\otimes M\to\mathbb{C}^2\otimes\mathbb{C}^2\otimes M'$. Then define $\psi^\heartsuit:=(\omega\otimes 1_M)\circ(1_{\mathbb{C}^2}\otimes\psi)\in\mathrm{Hom}_\Gamma(\mathbb{C}^2\otimes M,M')$. Conversely, we can map $\varphi\in\mathrm{Hom}_\Gamma(\mathbb{C}^2\otimes M,M')$ to $(1_{\mathbb{C}^2}\otimes\varphi)\circ(\zeta\otimes 1_M)\in\mathrm{Hom}(M,\mathbb{C}^2\otimes M')$.

Exercise 4.7. *Check that the maps* $\mathrm{Hom}(M,\mathbb{C}^2\otimes M')\to\mathrm{Hom}(\mathbb{C}^2\otimes M,M'),\psi\mapsto(\omega\otimes 1_M)\circ(1_{\mathbb{C}^2}\otimes\psi)$ *and* $\mathrm{Hom}(\mathbb{C}^2\otimes M,M')\to\mathrm{Hom}(M,\mathbb{C}^2\otimes M'),\varphi\mapsto(1_{\mathbb{C}^2}\otimes\varphi)\circ(\zeta\otimes 1_M)$ *are inverse to each other.*

The following claim is [CBH, Lemma 3.2].

Lemma 4.2. *To each $a\in\underline{Q}_1$ one can associate $\eta_a\in\mathrm{Hom}_\Gamma(N_{t(a)},\mathbb{C}^2\otimes N_{h(a)}),\theta_a\in\mathrm{Hom}_\Gamma(N_{h(a)},\mathbb{C}^2\otimes N_{t(a)})$ that combine to form bases in the spaces $\mathrm{Hom}_\Gamma(N_i,\mathbb{C}^2\otimes N_j)$ are all i,j and satisfy*

$$\sum_{a\in\underline{Q}_1,t(a)=i}(1_{\mathbb{C}^2}\otimes\theta_a)\eta_a-\sum_{a\in\underline{Q}_1,h(a)=i}(1_{\mathbb{C}^2}\otimes\eta_a)\theta_a=\delta_i(\zeta\otimes 1_{N_i}), \tag{2}$$

(the equality of maps $N_i \to \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes N_i$) for all i.

There are two rather different cases: Γ is cyclic or Γ is non-cyclic. The difference is that in the second case $\underline{Q}$ is a tree, while in the first case $\underline{Q}$ is not.

Problem 4.8. *Prove the CBH lemma in the cyclic case, assuming that the orientation on $\underline{Q}$ is also cyclic. Hint: for x, y we can take Γ-eigenvectors.*

The case when Γ is non-cyclic will be considered in the next lecture.

References

[CBH] W. Crawley-Boevey, M. Holland. *Noncommutative deformations of Kleinian singularities.* Duke Math. J. 92(1998), 605-635.