LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

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4. Deformed preprojective algebras, cont'd

4.1. **Recap.** Recall that in the previous lecture we have identified $\mathbb{C}\langle x,y\rangle\#\Gamma$ with $T_{\mathbb{C}\Gamma}(\mathbb{C}^2\otimes\mathbb{C}\Gamma)$, and $\mathbb{C}Q$ with $T_{(\mathbb{C}Q)^0}(\mathbb{C}Q)^1$. Also recall that $f\mathbb{C}\langle x,y\rangle\#\Gamma f\cong \mathbb{C}Q$, where $f=\bigoplus_{i=0}^r f_i\in \mathbb{C}\Gamma=\bigoplus_{i=0}^r \operatorname{End}(N_i^*)$ with f_i being a primitive idempotent in $\operatorname{End}(N_i)^*$. Under this identification, $f_i\in\mathbb{C}Q$ becomes the path ϵ_i.

Further, to $i\in Q_0$ we have assigned an element $[a^*,a]_i\in \epsilon_i(\mathbb{C}Q)^2\epsilon_i$ by the formula

$$[a^*,a]_i=\bigoplus_{a\in \underline{Q}_1, t(a)=i} a^*a-\sum_{a\in\underline{Q}_1,h(a)=i} aa^*.$$

Also to $c\in(\mathbb{C}\Gamma)^\Gamma$ we assign $\lambda=(\lambda_i)_{i\in Q_0}$ by $\lambda_i=\operatorname{tr}_{N_i}c$. The main result we are going to prove is a theorem of Crawley-Boevey and Holland.

Theorem 4.1. *The ideal $f(xy-yx-c)_{\mathbb{C}\langle x,y\rangle\#\Gamma}f$ is generated by the elements $[a^*,a]_i-\lambda_i\epsilon_i, i\in Q_0$.*

A key step in the proof is the following lemma again due to Crawley-Boevey and Holland.

Lemma 4.2. *To each $a\in\underline{Q}_1$ one can associate $\eta_a\in\operatorname{Hom}_\Gamma(N_{t(a)},\mathbb{C}^2\otimes N_{h(a)}), \theta_a\in \operatorname{Hom}_\Gamma(N_{h(a)},\mathbb{C}^2\otimes N_{t(a)})$ that combine to form bases in the spaces $\operatorname{Hom}_\Gamma(N_i,\mathbb{C}^2\otimes N_j)$ are all i,j and satisfy*

$$\sum_{a\in\underline{Q}_1,t(a)=i}(1_{\mathbb{C}^2}\otimes\theta_a)\eta_a-\sum_{a\in\underline{Q}_1,h(a)=i}(1_{\mathbb{C}^2}\otimes\eta_a)\theta_a=\delta_i(\zeta\otimes 1_{N_i}), \tag{1}$$

(the equality of maps $N_i\to\mathbb{C}^2\otimes\mathbb{C}^2\otimes N_i$) for all i.

To prove the lemma we have introduced explicit mutually inverse isomorphisms of the spaces $\operatorname{Hom}_\Gamma(M,\mathbb{C}^2\otimes M'),\operatorname{Hom}_\Gamma(\mathbb{C}^2\otimes M,M')$. Namely, we map $\psi\in\operatorname{Hom}_\Gamma(M,\mathbb{C}^2\otimes M')$ to $\psi^\heartsuit:=(\omega\otimes 1_{M'})\circ(1_{\mathbb{C}^2}\otimes\psi)\in\operatorname{Hom}_\Gamma(\mathbb{C}^2\otimes M,M')$, and we map $\varphi\in\operatorname{Hom}_\Gamma(\mathbb{C}^2\otimes M,M')$ to $(1_{\mathbb{C}^2}\otimes\varphi)\circ(\zeta\otimes 1_M)\in\operatorname{Hom}(M,\mathbb{C}^2\otimes M')$. Here ω is the skew-symmetric form on $\mathbb{C}^2$ given by $\omega(y,x)=1=-\omega(x,y)$ (and viewed as a map $\mathbb{C}^2\otimes\mathbb{C}^2\to\mathbb{C}$) and $\zeta=x\otimes y-y\otimes x$ (viewed as a map $\mathbb{C}\to\mathbb{C}^2\otimes\mathbb{C}^2$).

4.2. **Proof of the CBH lemma.** From now on we concentrate on the non-cyclic case. A special feature of this case is that Γ is a tree.

The spaces $\operatorname{Hom}_\Gamma(N_i,\mathbb{C}^2\otimes N_j)$ when $i=t(a),j=h(a)$ or vice versa are 1-dimensional. For a moment, choose arbitrary nonzero η_a,θ_a, they are defined up to a nonzero scalar multiple. Then $\theta_a^\heartsuit\eta_a$ is a nonzero endomorphism of $N_{t(a)}$, while $\eta_a^\heartsuit\theta_a$ is a nonzero endomorphism of $N_{h(a)}$. Multiplying θ_a by a nonzero scalar k, we also multiply those two endomorphisms by k. We claim that there are nonzero scalars $d_i, i\in Q_0$, with the property that (after rescaling the θ_i's) we get

$$\theta_a^\heartsuit\eta_a=d_{h(a)}1_{N_{t(a)}},\eta_a^\heartsuit\theta_a=-d_{t(a)}1_{N_{h(a)}}. \tag{2}$$

This is a consequence of $\underline{Q}$ being a tree. Namely, we fix all η_a and some vertex i. Pick d_i. This fixes θ_a for all a with $h(a) = i$ (from $\theta_a^\heartsuit \eta_a = d_{h(a)} 1_{N_{t(a)}}$) or $t(a) = i$ (from $\eta_a^\heartsuit \theta_a = -d_{t(a)} 1_{N_{h(a)}}$) and so also d_j for all vertices j connected to i (for example, if $h(a) = i$, then $d_{t(a)}$ is determined from $\eta_a^\heartsuit \theta_a = -d_{t(a)} 1_{N_{h(a)}}$, where we now know the left hand side). Then we proceed with i replaced by one of these j's. Since our graph is a tree, we see that every vertex appears only ones, and when our argument finishes, we get all d_i and all θ_a fixed.

The map $\eta_a \theta_a^\heartsuit : \mathbb{C}^2 \otimes N_{h(a)} \to \mathbb{C}^2 \otimes N_{h(a)}$ therefore equals $d_{h(a)} \pi_{N_t(a)}$, where $\pi_{N_{t(a)}}$ is the projection to the summand $N_{t(a)}$ in $\mathbb{C}^2 \otimes N_{h(a)}$. Similarly, $\theta_a \eta_a^\heartsuit = -d_{t(a)} \pi_{N_{h(a)}}$. The modules $N_{t(a)}$ with $h(a) = i$, and $N_{h(a)}$ with $t(a) = i$ is a complete list of the simple summands of $\mathbb{C}^2 \otimes N_i$. So for $i \in Q_0$ we have

$$\sum_{a \in \underline{Q}_1, h(a)=i} \eta_a \theta_a^\heartsuit - \sum_{a \in \underline{Q}_1, t(a)=i} \theta_a \eta_a^\heartsuit = d_i 1_{\mathbb{C}^2 \otimes N_i}. \tag{3}$$

We want to tensor the previous equation with $1_{\mathbb{C}^2}$ (on the left) and compose with $\zeta \otimes 1$ (on the right) so that we get an equality of maps $N_i \to \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes N_i$. We have $(1_{\mathbb{C}^2} \otimes \theta_a \eta_a^\heartsuit) \circ (\zeta \otimes 1) = (1_{\mathbb{C}^2} \otimes \theta_a)(1_{\mathbb{C}^2} \otimes \eta_a^\heartsuit)(\zeta \otimes 1_{N_i}) = (1_{\mathbb{C}^2} \otimes \theta_a)\eta_a$. So what we get is

$$\sum_{a \in \underline{Q}_1, h(a)=i} (1_{\mathbb{C}^2} \otimes \eta_a)\theta_a - \sum_{a \in \underline{Q}_1, t(a)=i} (1_{\mathbb{C}^2} \otimes \theta_a)\eta_a = d_i \zeta \otimes 1_{N_i}.$$

To show that we can take $-\delta_i$ for d_i, we compose both sides of the equality on the left with $\omega \otimes 1$ (to get a map $N_i \to N_i$). We have $(\omega \otimes 1_{N_i})(1_{\mathbb{C}^2} \otimes \theta_a)\eta_a = \theta_a^\heartsuit \eta_a = d_{h(a)} 1_{N_i}$ and similarly $(\omega \otimes 1_{N_i})(1_{\mathbb{C}^2} \otimes \eta_a)\theta_a = -d_{t(a)} 1_{N_i}$. So on the left hand side we get $-\sum_j d_j 1_{N_i}$, where we sum over all j connected to i. On the right hand side we get $-\omega(\zeta) d_i 1_{N_i} = -2 d_i 1_{N_i}$. So $2d_i - \sum_j d_j = 0$ for each $i \in Q_0$. This is a linear system whose matrix is precisely the Cartan matrix of the extended Dynkin diagram. The space of solutions of this equation is 1-dimensional and is generated by δ. Rescaling the maps θ_a, we achieve $d_i = -\delta_i$.

4.3. **Proof of Theorem 4.1.** We need to interpret (1) so that it becomes an equality in $\mathbb{C}Q = T_{(\mathbb{C}Q)^0}(\mathbb{C}Q)^1 = f\mathbb{C}\langle x, y\rangle \# \Gamma f$.

$\mathrm{Hom}_\Gamma(N_i, \mathbb{C}^2 \otimes N_j)$ is just $\mathrm{Hom}_\Gamma(\mathbb{C}\Gamma f_i, \mathbb{C}^2 \otimes \mathbb{C}\Gamma f_j) = f_j(\mathbb{C}^2 \otimes \mathbb{C}\Gamma) f_i = \epsilon_j (\mathbb{C}Q)^1 \epsilon_i$. We take η_a for a, and θ_a for a^*. Then, similarly, $\mathrm{Hom}_\Gamma(N_i, \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes N_i) = \mathrm{Hom}_\Gamma(\mathbb{C}\Gamma f_i, \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}\Gamma f_j) = f_j(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}\Gamma) f_i = \epsilon_i (\mathbb{C}Q)^2 \epsilon_i$, and $(1 \otimes \theta_a)\eta_a \in \mathrm{Hom}_\Gamma(N_i, \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes N_i)$ is nothing else but $a^* a$.

The elements $xy - yx$ and c of $\mathbb{C}\langle x, y\rangle \# \Gamma$ are Γ-invariant and hence commute with $\mathbb{C}\Gamma$ and in particular, with the idempotents f_i. Under our identifications, the element $(xy - yx) f_i = f_i (xy - yx) f_i$ is nothing else but $\zeta \otimes 1_{N_i}$. So the CBH Lemma just says that

$$\sum_{a \in \underline{Q}_1, t(a)=i} a^* a - \sum_{a \in \underline{Q}_1, h(a)=i} a a^* = \delta_i (xy - yx) f_i$$

We remark that $\delta_i c f_i$ is precisely $\lambda_i f_i$, so

$$\sum_{a \in \underline{Q}_1, t(a)=i} a^* a - \sum_{a \in \underline{Q}_1, h(a)=i} a a^* - \lambda_i f_i = \delta_i (xy - yx - c) f_i$$

Now we notice that the left hand sides of the previous equality all lie in $f(xy - yx - c)_{\mathbb{C}\langle x,y\rangle \# \Gamma} f$. On the other hand, let us show that $f(xy - yx - c)_{\mathbb{C}\langle x,y\rangle \# \Gamma} f$ coincides with the ideal in $\mathbb{C}Q$ generated by $(xy - yx - c) f$. Recall that $\mathbb{C}\Gamma f \mathbb{C}\Gamma = \mathbb{C}\Gamma$ and so there are elements $r_j, s_j \in \mathbb{C}\Gamma$ with $\sum_j r_j f s_j = 1$. So $\sum_j r_j (xy - yx - c) f s_j = \sum_j (xy - yx - c) r_j f s_j = xy - yx - c$.

So $xy - yx - c$ lies in the ideal of $\mathbb{C}\langle x, y\rangle\#\Gamma$ generated by $(xy - yx - c)f$. It follows that $(xy - yx - c)_{\mathbb{C}\langle x,y\rangle\#\Gamma} \cap f(\mathbb{C}\langle x, y\rangle\#\Gamma)f = ((xy - yx - c)f)_{\mathbb{C}Q}$.

4.4. **Remarks and ramifications.**

4.4.1. *Orientation.* Formally, the algebra Π^λ (and the map $\mu : \mathrm{Rep}(Q, \delta) \to \mathfrak{gl}(\delta)$) depends on the orientation. Pick an arrow $b \in \underline{Q}_1$ and switch its orientation, let b_1 denote the same arrow with the inverted orientation. Let Π'^λ, μ' be constructed from this new orientation. The map $\epsilon_i \mapsto \epsilon_i, a \mapsto a, a^* \mapsto a^*, a \in \underline{Q}_1 \setminus \{b\}, b \mapsto b_1^*, b^* \mapsto -b_1$ extends to an automorphism of $\mathbb{C}Q$ that gives rise to an algebra isomorphism $\Pi^\lambda \mapsto \Pi'^\lambda$. Also this automorphism gives rise to a linear $\mathrm{GL}(\delta)$-equivariant automorphism of $\mathrm{Rep}(Q, \delta)$ that intertwines the maps μ and μ'. So our constructions do not depend on the choice of an orientation up to distinguished isomorphisms.

4.4.2. *Identification of* $\mathrm{Rep}_\Gamma(\mathbb{C}[x, y]\#\Gamma, \mathbb{C}\Gamma)/\!/ \mathrm{GL}(\mathbb{C}\Gamma)^\Gamma$ *with* $\mathbb{C}^2/\Gamma$. We have identified

$$\mathrm{Rep}_\Gamma(\mathbb{C}[x, y]\#\Gamma, \mathbb{C}\Gamma)/\!/ \mathrm{GL}(\mathbb{C}\Gamma)^\Gamma$$

(viewed as a set of isomorphism classes of semisimple representations) with $\mathbb{C}^2/\Gamma$. Let us show that this is an identification of algebraic varieties. Recall the spherical subalgebra $\mathbb{C}[x, y]^\Gamma \cong e(\mathbb{C}[x, y]\#\Gamma)e \subset \mathbb{C}[x, y]\#\Gamma$. An element $\varphi \in \mathrm{Rep}_\Gamma(\mathbb{C}[x, y]\#\Gamma, \mathbb{C}\Gamma)$ restricts to a representation of $e(\mathbb{C}[x, y]\#\Gamma)e$ in $e\mathbb{C}\Gamma = \mathbb{C}$. The latter is nothing else but a point of $\mathbb{C}^2/\Gamma$ and so we get another map $\xi : \mathrm{Rep}_\Gamma(\mathbb{C}[x, y]\#\Gamma, \mathbb{C}\Gamma) \to \mathbb{C}^2/\Gamma$ that is clearly $\mathrm{GL}(\mathbb{C}\Gamma)^\Gamma$-equivariant and so descends to $\mathrm{Rep}_\Gamma(\mathbb{C}[x, y]\#\Gamma, \mathbb{C}\Gamma)/\!/ \mathrm{GL}(\mathbb{C}\Gamma)^\Gamma \to \mathbb{C}^2/\Gamma$. It is clear from the construction that it coincides with our previous map, given by taking the central character. Our new map is a morphism of algebraic varieties: the pull-back $\xi^*(F)$ of $F \in \mathbb{C}[x, y]^\Gamma$ evaluated on a representation φ is just a matrix coefficient of φ evaluated on eFe (or F) and so $\xi^*(F)$ is a polynomial on $\mathrm{Rep}_\Gamma(\mathbb{C}[x, y]\#\Gamma, \mathbb{C}\Gamma)$.

4.4.3. *Scheme structure on* $\mu^{-1}(0)/\!/G$. Before we have viewed $\mu^{-1}(0)/\!/G$ as a variety, but in fact, it is an affine scheme with the following algebra of functions: $(\mathbb{C}[R]/\mathbb{C}[R]\mu^*(\mathfrak{g}))^G$, where we write G for $\mathrm{GL}(\delta)$, $\mathfrak{g}$ for $\mathfrak{gl}(\delta)$ and R for $\mathrm{Rep}(Q, \delta)$. Here $\mu^* : \mathfrak{g} \to \mathbb{C}[R]$ is a pull-back map induced by $\mu : R \to \mathfrak{g}$ (we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the trace pairings on all $\mathfrak{gl}_{\delta_i}(\mathbb{C})$).

It turns out however, that even the subscheme $\mu^{-1}(0) \subset R$ (not only the quotient $\mu^{-1}(0)/\!/G$) is reduced (and is a complete intersection). This follows from the claim that every irreducible component of $\mu^{-1}(0)$ contains a G-orbit with trivial stabilizer (and also from some standard properties of moment maps). The claim about the existence of an orbit follows from the representation theory of quivers.

Problem 4.1. *Check the claims of the previous paragraph by hand in the case of the cyclic quiver* Q.

4.4.4. $\mathrm{Rep}_\Gamma(H_c, \mathbb{C}\Gamma)$. We have identified $\mu^{-1}(0)/\!/ \mathrm{GL}(\delta)$ with $\mathrm{Rep}_\Gamma(H_c, \mathbb{C}\Gamma)/\!/ \mathrm{GL}(\mathbb{C}\Gamma)^\Gamma$ using Theorem 4.1 with $c = 0$. Let us see what happens for arbitrary c.

First of all, we claim that H_c has no representations in $\mathbb{C}\Gamma$ if $c_1 \neq 0$. Indeed, $xy - yx$ acts in the same way as c. In particular, $c \in (\mathbb{C}\Gamma)^\Gamma$ has trace 0 on any H_c-module. But the trace of c on $\mathbb{C}\Gamma$ is $|\Gamma|c_1$. From now on, we consider the case $c_1 = 0$.

Thanks to Theorem 4.1, $\mu^{-1}(\lambda)/\!/ \mathrm{GL}(\delta)$ is identified with $\mathrm{Rep}_\Gamma(H_c, \mathbb{C}\Gamma)/\!/ \mathrm{GL}(\mathbb{C}\Gamma)^\Gamma$. Similarly to a remark above, we have a morphism $\mathrm{Rep}_\Gamma(H_c, \mathbb{C}\Gamma)/\!/\mathbb{C}\Gamma \to \mathrm{Rep}(eH_ce, \mathbb{C})$. We will see below that if $c_1 = 0$, then the algebra eH_ce is commutative. So $\mathrm{Rep}(eH_ce, \mathbb{C}) = \mathrm{Spec}(eH_ce)$. Also one can show that the morphism $\mathrm{Rep}_\Gamma(H_c, \mathbb{C}\Gamma)/\!/\mathbb{C}\Gamma \to \mathrm{Rep}(eH_ce, \mathbb{C})$ is an isomorphism. We are not going to do this, but we will prove that $\mu^{-1}(\lambda)/\!/\operatorname{GL}(\delta) \cong \operatorname{Spec}(eH_{c'}e)$ (perhaps for a different c'). Also we will see that the map $z \mapsto ez$ from the center $Z(H_c)$ of H_c to eH_ce is an isomorphism of algebras.

4.4.5. *Deformations of* $\mathbb{C}^2/\Gamma$. Finally, let us remark that the algebras $\mathbb{C}[\mu^{-1}(\lambda)/\!/G]$ with $\sum_{i=0}^r \delta_i\lambda_i = 0$ (this is equivalent to $c_1 = 0$) form an r-parametric deformation of $\mathbb{C}^2/\Gamma$. Of course, all deformations obtained in this way are commutative. We will see below that one can produce non-commutative using a related construction called a *quantum Hamiltonian reduction.*

5. Symplectic quotient singularities

5.1. **Quotient singularities.** We are interested in studying deformations of the algebras of the form $S(V)^\Gamma$, where V is a vector space, $S(V)$ is its symmetric algebra, and Γ is a finite subgroup of $\operatorname{GL}(V)$. This algebra is the algebra of polynomial functions on the quotient V^*/Γ. In fact, to get some non-trivial theory we will need to restrict the class of groups we are dealing with. Before we considered $\Gamma \subset \operatorname{SL}_2(\mathbb{C})$ so our first guess would be that we need $\Gamma \subset \operatorname{SL}(V)$. However, this class is still too large. We will consider the case when V is a symplectic vector space, i.e., possesses a non-degenerate skew-symmetric form, say ω, and Γ preserves this form, i.e., lies in the symplectic group $\operatorname{Sp}(V)$.

5.2. **Poisson brackets.** As usual, one of the reasons why we make this restriction is that this situation is easier. But there is a reason for that too. Roughly speaking, a (non-commutative) deformation of a commutative algebra gives rise to a new structure on this algebra, a *Poisson bracket*, and, for $\Gamma \subset \operatorname{Sp}(V)$, the algebra $S(V)^\Gamma$ already comes equipped with such a bracket.

Let us start with a general definition of a (Poisson bracket). Let A be a commutative associative unital algebra. A *bracket* on A is a skew-symmetric $\mathbb{C}$-bilinear map $\{\cdot,\cdot\} : A\times A \to A$ satisfying the following two axioms, known as the Leibniz identity:

(L) $\{a, bc\} = \{a, b\}c + \{a, c\}b$,

for all $a, b, c \in A$. We remark, that thanks to $\{\cdot,\cdot\}$ being skew-symmetric, we have also $\{ab, c\} = a\{b, c\} + b\{a, c\}$. A bracket is called *Poisson* if it satisfies the Jacobi identity

(J) $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$,

An algebra equipped with a Poisson bracket is called a Poisson algebra.

Exercise 5.1. *Let A be a commutative associative unital algebra.*

(1) *Let A be equipped with a bracket $\{\cdot,\cdot\}$. Show that $\{1, a\} = 0$ for all $a \in A$.*
(2) *Show that if $a_1, \ldots, a_k$ are generators of A, then there is at most one bracket $\{\cdot,\cdot\}$ with given $\{a_i, a_j\}$. Show that this bracket satisfies the Jacobi identity for all a, b, c, if it does so for all a_i, a_j, a_k.*
(3) *Finally, prove that if $A = \mathbb{C}[a_1, \ldots, a_k]$, then a bracket exists for any values of $\{a_i, a_j\}$ as long as $\{a_i, a_j\} = -\{a_j, a_i\}$.*

Let us proceed to examples. Let V, ω and Γ be as above. Define $\{\cdot,\cdot\}$ on $S(V)$ by setting $\{u, v\} := \omega(u, v)$ for $u, v \in V$ and extending this bracket to the whole $S(V)$ in a unique possible way. By the previous exercise we get a Poisson bracket, since $\{\{u, v\}, w\} = 0$ for all $u, v, w \in V$.

Exercise 5.2. *We can choose a basis* $x_1, \ldots, x_n, y_1, \ldots, y_n$ *in* V *so that* $\omega(x_i, x_j) = \omega(y_i, y_j) = 0, \omega(y_i, x_j) = \delta_{ij}$. *Let us identify* $S(V)$ *with* $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. *Then* $\{\cdot, \cdot\}$ *is given by the formula*

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i}.$$

Now, since $\Gamma \subset \mathrm{Sp}(V)$ we have $\omega(\gamma u, \gamma v) = \omega(u, v)$ and hence $\{\gamma u, \gamma v\} = \{u, v\}$. We deduce that γ leaves the bracket on $S(V)$ invariant, i.e., $\{\gamma f, \gamma g\} = \gamma\{f, g\}$ for all $\gamma \in \Gamma, f, g \in S(V)$. In particular, the subalgebra of invariants $S(V)^\Gamma$ is closed under $\{\cdot, \cdot\}$ and so it becomes a Poisson algebra.

Let us make a remark regarding a compatibility between the brackets and gradings. Assume that A is graded, $A = \bigoplus_{n=0}^{\infty} A^n$. We say that $\{\cdot, \cdot\}$ has degree $-d$ if $\{A^i, A^j\} \subset A^{i+j-d}$. For example, the Poisson bracket on $S(V)$ (and hence also on $S(V)^\Gamma$ has degree -2).

Finally, let us discuss a connection between Poisson brackets and (filtered) deformations. Let $\mathcal{A}$ be a filtered (associative unital) algebra. Assume that $A := \operatorname{gr}\mathcal{A}$ is commutative. This means that $[\mathcal{A}^{\leqslant i}, \mathcal{A}^{\leqslant j}] \subset \mathcal{A}^{i+j-1}$. Let us pick a positive integer d such that $[\mathcal{A}^{\leqslant i}, \mathcal{A}^{\leqslant j}] \subset \mathcal{A}^{i+j-d}$. We can define a bracket of degree $-d$ on A in a way similar to the definition of the product. Namely, pick $a \in A^i, b \in A^j$ and lift them to elements $\bar{a} \in \mathcal{A}^{\leqslant i}, \bar{b} \in \mathcal{A}^{\leqslant j}$. Then set $\{a, b\} := [\bar{a}, \bar{b}] + \mathcal{A}^{\leqslant i+j-d-1}$, this is an element of A^{i+j-d}.

Exercise 5.3. *Check that* $\{\cdot, \cdot\}$ *on* $A = \operatorname{gr}\mathcal{A}$ *is well-defined and is indeed a Poisson bracket.*