LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

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6. Symplectic reflection algebras

6.1. **Definition of SRA.** Let V be a finite dimensional complex vector space equipped with a non-degenerate skew-symmetric form ω. Then there is a distinguished filtered deformation of the symmetric algebra $S(V)$. This is the Weyl algebra $W(V) = T(V)/(u\otimes v - v\otimes u - \omega(u,v))$.

Exercise 6.1. *Show that $W(V)$ is a filtered deformation of $S(V)$ (the case $\dim V = 2$ was considered above). Moreover, check that the Poisson bracket on $S(V)$ induced from $W(V)$ coincides with the initial bracket.*

Let Γ be a finite subgroup of $\mathrm{Sp}(V)$. We want to study filtered deformations of $S(V)^\Gamma$ that are compatible with the Poisson bracket on $S(V)^\Gamma$. For non-commutative deformations, this means that the bracket on $S(V)^\Gamma$ induced by the deformation has to coincide with (or be proportional to) the initial bracket on $S(V)^\Gamma$. One can state a compatibility condition for commutative deformations as well (they itself have to be Poisson algebras) but we are not going to do this.

As before, we are going to produce deformations of $S(V)\#\Gamma$ first (and then pass to spherical subalgebras to get deformations of $S(V)^\Gamma$, we will recall how this is done below). As in the case when $\dim V = 2$, we have $S(V)\#\Gamma = T(V)\#\Gamma/(u\otimes v - v\otimes u \,|\, u,v\in V)$. So we can take a linear map $\kappa : \bigwedge^2 V \to (T(V)\#\Gamma)^{\leqslant 1}$ and form the quotient $H_\kappa = T(V)\#\Gamma/(u\otimes v - v\otimes u - \kappa(u,v))$. Of course, $H_0 = S(V)\#\Gamma$.

Exercise 6.2. *Show that if $\operatorname{gr} H_\kappa = S(V)\#\Gamma$, then κ is a Γ-equivariant map (where Γ acts on $\mathbb{C}\Gamma$ via the adjoint representation). Furthermore, show that if $-1_V \in \Gamma$, then the image of κ lies in $\mathbb{C}\Gamma$.*

In general, it is still a good idea to consider only $\kappa : \bigwedge^2 V \to \mathbb{C}\Gamma$. This is motivated, in part, by our compatibility condition of Poisson brackets: we want filtered deformations $\mathcal{A}$ of $S(V)^\Gamma$ with $[\mathcal{A}^{\leqslant i}, \mathcal{A}^{\leqslant j}] \subset \mathcal{A}^{i+j-2}$. Of course, H_κ deforms $S(V)\#\Gamma$, not $S(V)^\Gamma$, but it is still reasonable to require that $[u,v] \in H_\kappa^{\leqslant 0}$ (that should be equal to $\mathbb{C}\Gamma$). So below we only consider Γ-equivariant κ with image in $\mathbb{C}\Gamma$. We can write κ as $\sum_{\gamma\in\Gamma}\kappa_\gamma\gamma$, where $\kappa_\gamma \in \bigwedge^2 V^*$. It turns out that for $\operatorname{gr} H_\kappa = S(V)\#\Gamma$ some κ_γ must vanish.

We map V to H_κ via $V \hookrightarrow T(V)\#\Gamma \twoheadrightarrow H_\kappa$, this is an embedding whenever $\operatorname{gr} H_\kappa = S(V)\#\Gamma$. In H_κ we must have $[[u,v],w] + [[v,w],u] + [[w,u],v] = 0$ for all $u,v,w\in V$, equivalently,

$$[\kappa(u,v),w] + [\kappa(v,w),u] + [\kappa(w,u),v] = 0. \tag{1}$$

Exercise 6.3. *We have $[\kappa(u,v),w] = \sum_{\gamma\in\Gamma}\kappa_\gamma(u,v)(\gamma(w)-w)\gamma$.*

Since $\operatorname{gr} H_\kappa = S(V)\#\Gamma$, the map $V\otimes\mathbb{C}\Gamma \to H_\kappa$ induced by the embedding $V\otimes\mathbb{C}\Gamma \to T(V)\#\Gamma$ is injective. It follows that the equalities

$$\kappa_\gamma(u,v)(\gamma(w)-w) + \kappa_\gamma(v,w)(\gamma(u)-u) + \kappa_\gamma(w,u)(\gamma(v)-v) = 0 \tag{2}$$

hold for any $\gamma \in \Gamma$. Let us show that if $\mathrm{rk}(\gamma - 1_V) > 2$, then $\kappa_\gamma = 0$. Indeed, if u, v, w are such that $\gamma(u) - u, \gamma(v) - v, \gamma(w) - w$ are linearly independent, then we must have $\kappa_\gamma(u, v) = \kappa_\gamma(v, w) = \kappa_\gamma(w, u) = 0$. But the linear independence definitely holds for vectors in general position provided $\mathrm{rk}(\gamma - 1_V) > 2$, so κ is 0.

Exercise 6.4. *Show that* $\mathrm{im}(\gamma - 1_V) \oplus \ker(\gamma - 1_V) = V$ *for any* $\gamma \in \Gamma$. *Further, show that these space are orthogonal with respect to* ω *and, in particular, the restrictions of* ω *to these subspaces are non-degenerate.*

Now consider the case when $\mathrm{rk}(\gamma - 1_V) = 2$. Suppose that in (2) $\gamma(w) = w$. Then $\kappa_\gamma(w, v)(\gamma(u) - u) = \kappa_\gamma(w, u)(\gamma(v) - v)$. If $\gamma(u) - u, \gamma(v) - v$ are linearly independent (that is true for u, v in general position), then $\kappa_\gamma(w, v) = 0$. It follows that κ_γ is proportional, say with coefficient c_γ, to the form ω_γ defined by $\omega_\gamma = \omega$ on $\mathrm{im}(\gamma - 1_V)$ and $\ker(\gamma - 1_V) \subset \ker \omega_\gamma$.

Exercise 6.5. *Show that* $\omega_\gamma(u, v)(\gamma(w) - w) + \omega_\gamma(v, w)(\gamma(u) - u) + \omega_\gamma(w, u)(\gamma(v) - v) = 0$.

Finally, let us consider κ_1. This is a Γ-invariant skew-symmetric form on V. We say that V is *symplectically irreducible* if there is no Γ-invariant symplectic subspace, i.e. a subspace with non-degenerate restriction of ω.

Exercise 6.6. *Show that a symplectically irreducible* Γ*-module* V *is either irreducible, or is the sum* $U \oplus U^*$*, where* U *is irreducible and not symplectic. Deduce that the space of* Γ*-invariant skew-symmetric forms on* V *is one-dimensional and so is generated by* ω.

So we have $\kappa(u, v) = t\omega(u, v) + \sum_{s \in S} c_s \omega_s(u, v)s$, where S is the set of all $s \in \Gamma$ with $\mathrm{rk}(\gamma - 1_V) = 2$, such s are called *symplectic reflections*, we will later explain a reason for this name. We remark that S is a union of conjugacy classes.

We should have $\kappa(\gamma u, \gamma v) = \gamma \kappa(u, v) \gamma^{-1}$, equivalently,

$$\sum_{s \in S} c_s \omega_s(\gamma u, \gamma v)s = \sum_{s \in S} c_s \omega_s(u, v)\gamma s \gamma^{-1} = \sum_{s \in S} c_{\gamma^{-1} s \gamma} \omega_{\gamma^{-1} s \gamma}(u, v)s.$$

It is straightforward to see that $\omega_{\gamma^{-1} s \gamma}(u, v) = \omega_s(\gamma u, \gamma v)$ – both forms are just projections of ω to $\mathrm{im}(\gamma^{-1} s \gamma - 1_V)$. So the condition that κ is Γ-equivariant translates to $c_s = c_{\gamma^{-1} s \gamma}$ for all $\gamma \in \Gamma, s \in S$, i.e., the map $s \mapsto c_s$ has to be Γ-invariant.

Now we are ready to give the definition of a Symplectic reflection algebra due to Etingof and Ginzburg, [EG]. Take $\Gamma \subset \mathrm{Sp}(V)$ (we do not require now that V is symplectically irreducible). Pick a complex number t and a conjugation invariant function $c \mapsto c_s : S \to \mathbb{C}$. Then set

$$H_{t,c} = T(V)\#\Gamma / \left(u \otimes v - v \otimes u - t\omega(u, v) - \sum_{s \in S} c_s \omega_s(u, v)s \right).$$

This is a SRA. We will see below that $\mathrm{gr}\, H_{t,c}$ is indeed $S(V)\#\Gamma$.

6.2. **Symplectic reflection groups.** In the definition of a SRA only symplectic reflections matter. More precisely, consider the subgroup Γ' of Γ generated by the symplectic reflections. This is a normal subgroup. Let $H'_{t,c}$ be the SRA defined for Γ'. Then $H_{t,c}$ is naturally identified with the algebra $H'_{t,c}\#_{\Gamma'}\Gamma$ defined as follows: as a vector space $H'_{t,c}\#_{\Gamma'}\Gamma$ is $H'_{t,c} \otimes_{\mathbb{C}\Gamma'} \mathbb{C}\Gamma$, where we view $\mathbb{C}\Gamma$ as a $\mathbb{C}\Gamma'$-module with respect to the left action of Γ' and $H'_{t,c}$ with respect to the right action. The algebra structure is introduced by the analogy with the usual smash-product.

This is why people usually consider the SRA only for *symplectic reflection groups*, i.e., groups generated by symplectic reflections. The full classification of such groups is known, see [C]. We will need only two classes of such groups.

First, we remark that any non-unit element of Γ is a symplectic reflection provided $\dim V = 2$. So any Kleinian group is a symplectic reflection group. This example has a higher dimensional generalization: wreath-product groups. Namely, let $L = \mathbb{C}^2$, with a symplectic form ω_L. Set $V := L^{\oplus n}, \omega = \omega_L^{\oplus n}$. Pick a Kleinian group Γ_1. For $\Gamma(= \Gamma_n)$ we take the semi-direct product $\mathfrak{S}_n \ltimes \Gamma_1^n$. It acts on V as follows. Let $\gamma_{(i)}$ denote the element $\gamma \in \Gamma_1$ in the ith copy of Γ_1. It acts as γ on the ith copy of L and as 1 on the other copies. The subgroup $\mathfrak{S}_n \subset \Gamma$ acts by permuting the copies of L. It is clear that this group preserves ω.

Let us describe the symplectic reflections in Γ_n. First of all, any $\gamma_{(i)}$ is a symplectic reflection. Clearly, $\gamma_{(i)}$ and $\gamma'_{(j)}$ are conjugate if and only if γ and γ' are conjugate in Γ_1. So we have r conjugacy classes $S_1, \ldots, S_r$ of symplectic reflection in Γ_n, one per non-trivial conjugacy class in Γ_1. For $n > 1$, there is yet another class of symplectic reflections: it contains a transposition s_{ij} from $\mathfrak{S}_n$. This class, S_0, consists of the elements of the form $s_{ij}\gamma_{(i)}\gamma_{(j)}^{-1}$, where s_{ij} is the transposition in $\mathfrak{S}_n$ permuting the elements with indexes i and j.

Exercise 6.7. *Prove that $S_0, \ldots, S_r$ exhaust the conjugacy classes of symplectic reflections in Γ_n.*

In any case, we see that Γ_n is generated by symplectic reflections. Also it is clear that V is symplectically irreducible.

Another family we are going to consider is obtained from complex reflection groups. Namely, recall that by a *complex reflection* in the group $\mathrm{GL}(\mathfrak{h})$, where $\mathfrak{h}$ is a finite dimensional vector space, we mean an element s of finite order such that $\mathrm{rk}(s - 1_{\mathfrak{h}}) = 1$. Complex reflection groups (=finite groups generated by complex reflections) were classified in [ST]. They include all real reflection groups and, in particular, all Weyl groups.

Problem 6.8. *The goal of this problem is to construct a family of complex reflection groups, that includes Weyl groups of types B, D. Namely, fix $n, \ell \geqslant 1$ and a divisor r of ℓ. Consider all $n \times n$-matrices with the following properties: each column and each row contains a single non-zero element that is a root of 1 of order ℓ, and the product of the nonzero elements is a root of 1 of order r. Show that this is a complex reflection group. This group is denoted by $G(\ell, r, n)$. In particular, $B_n = G(2, 1, n)$ and $D_n = G(2, 2, n)$.*

In fact, there are only finitely many complex reflection groups different from $G(\ell, r, n)$.

Problem 6.9. *This problem concerns one exceptional complex reflection group, G_4. Take the Kleinian group Γ of type E_6. It has three two-dimensional irreducible representations, $\mathbb{C}^2$ and two other. Prove that the other two are dual to each other and Γ acts on them as a complex reflection group.*

Now let us explain how to produce a symplectic reflection group from a complex reflection group W (this construction is a kind of a justification of the name "symplectic reflection"). Set $V := \mathfrak{h} \oplus \mathfrak{h}^*$. We have a natural symplectic form on V: $\omega(\alpha, \beta) = \omega(a, b) = 0$ for $\alpha, \beta \in \mathfrak{h}^*, a, b \in \mathfrak{h}$, while $\omega(a, \alpha) = \langle a, \alpha \rangle$. For Γ we take the image of W in $\mathrm{GL}(\mathfrak{h}) \times \mathrm{GL}(\mathfrak{h}^*)$, i.e., $w(a, \alpha) = (wa, w\alpha)$ for $w \in W, a \in \mathfrak{h}, \alpha \in \mathfrak{h}^*$. Clearly, Γ preserves ω. An element $s \in \Gamma$ is a symplectic reflection if and only if the same element is a complex reflection in W.

The SRA corresponding to W has a special name, a Rational Cherednik algebra, shortly RCA (in the case when W is a Weyl group, this algebra is a "rational degeneration" of a certain algebra, a double affine Hecke algebra, introduced by Cherednik).

Problem 6.10. *Show that the relations for a RCA can be written as follows. For a complex reflection s let $\alpha_s \in \mathrm{im}(s-1_{\mathfrak{h}^*}), \alpha_s^\vee \in \mathrm{im}(s-1_{\mathfrak{h}})$ be such that $\langle \alpha_s, \alpha_s^\vee\rangle = 2$ (this is motivated by the Weyl group case). Show that the relations for the RCA can be written as*

$$[x,x']=0,[y,y']=0,[y,x]=t\langle y,x\rangle - \sum_{s\in S} c_s\langle \alpha_s, y\rangle\langle \alpha_s^\vee, x\rangle s, \quad x,x'\in\mathfrak{h}^*, y,y'\in\mathfrak{h}.$$

Note that here the coefficients c_s are not quite the same as in the presentation of an SRA. How are the coefficients related?

We remark that the two classes intersect. The intersection is the family $G(\ell,1,n) = (\mathbb{Z}/\ell\mathbb{Z})_n$. Formally, here we can also take $\ell=1$, the corresponding group is S_n. This case is somewhat degenerate, in particular, this group is not symplectically irreducible (when $\mathfrak{h}=\mathbb{C}^n$).

6.3. **Universal deformation.** In our proof of $\mathrm{gr}\, H_{t,c} = S(V)\#\Gamma$, we will use a different formalism. Namely, we will consider the universal SRA H obtained as follows. Let $S_1,\ldots,S_m$ be all classes of symplectic reflections in Γ. Pick independent variables $c_1,\ldots,c_m$, one for each conjugacy class S_i, and also an independent variable t. Consider the vector space P with basis $t,c_1,\ldots,c_m$. Then H will be the algebra over $S(P)$ defined by the same generators and relations as the usual SRA, i.e.,

$$H = S(P)\otimes T(V)\#\Gamma/(u\otimes v - v\otimes u - t\omega(u,v) - \sum_{i=1}^m c_i \sum_{s\in S_i}\omega_s(u,v)s).$$

An advantage of this setting is that the algebra H is graded: with Γ in degree 0, V in degree 1, and the parameter space P in degree 2. All algebras $H_{t',c'}$ for numerical t',c' have the form $\mathbb{C}\otimes_{S(P)} H$, where the epimorphism $S(P)\twoheadrightarrow\mathbb{C}$ given by $t\mapsto t', c_i\mapsto c_i'$. Also the filtration on $H_{t',c'}$ is induced from the grading on H.

Now let us define the notion of a *graded deformation* (this is a special case of the general notion of a deformation). Let $A=\bigoplus_{i=0}^\infty A^i$ be a graded commutative algebra with $A^0=\mathbb{C}$ and $\mathfrak{m}$ be the augmentation ideal $\bigoplus_{i=1}^\infty A^i$. Let B be a graded A-algebra (i.e., we have a graded algebra homomorphism $A\to B$). Set $B_0 = B/B\mathfrak{m}$. We say that B is a graded deformation of B_0 (over A), if B is a free A-module (this is, in fact, equivalent to flatness in this setting).

Exercise 6.11. *Suppose B is a graded deformation of B_0 over A. Show that a basis in B can be obtained as follows. Let ι be any graded section of the projection $B\twoheadrightarrow B_0$. Then for a basis of B (viewed as an A-module), we can take $\iota(B_0)$.*

We will see that H is a graded deformation of A. Moreover, we will see that (under the restriction that V is symplectically irreducible) H satisfies a certain universality property, a precise statement will be given below.

To prove this claim we will use a usual apparatus in the deformation theory, Hochschild cohomology. We will introduce them today. In the next lecture we will explain their connection to the deformation theory and then compute the relevant cohomology of $S(V)\#\Gamma$. With this computation, to deduce the claims of the previous paragraph will be relatively easy. Our main reference for Hochschild cohomology will be [E].

6.4. **Hochshild cohomology.** Let A be an associative algebra with unit and let M be an A-bimodule. By a Hochschild n-cochain with coefficients in M we mean a $\mathbb{C}$-linear map $A^{\otimes n} \to M$. The space of such cochains is denoted by $C^n(A,M)$. There is a map $d : C^n(A,M) \to C^{n+1}(A,M)$ given by

$$df(a_1 \otimes a_2 \otimes \ldots \otimes a_{n+1}) = f(a_1 \otimes \ldots \otimes a_n)a_{n+1} - f(a_1 \otimes \ldots \otimes a_{n-1} \otimes a_n a_{n+1}) + \\ + f(a_1 \otimes \ldots \otimes a_{n-1}a_n \otimes a_{n+1}) - \ldots + (-1)^n f(a_1a_2 \otimes a_3 \ldots \otimes a_{n+1}) + \\ + (-1)^{n+1} a_1 f(a_2 \otimes a_3 \otimes \ldots \otimes a_{n+1}).$$

Exercise 6.12. *Check that* $d^2 = 0$.

So the cochains $C^n(A,M)$ form a complex $C^0(A,M) \xrightarrow{d} C^1(A,M) \xrightarrow{d} C^2(A,M) \xrightarrow{d} \ldots$ called (not surprisingly) the Hochschild complex. Its cohomology, denoted by $\mathrm{HH}^i(A,M)$ (or simply $\mathrm{HH}^i(A)$ if $A = M$), are called the Hochshild cohomology.

Exercise 6.13. *Show that* $\mathrm{HH}^0(A,M)$ *coincides with the center of* M*, i.e., the space of all elements* $m \in M$ *such that* $am = ma$*. Show that Hochschild* 1*-cocycles are the derivations of* M*, i.e., the maps* $A \to M$ *that satisfy the Leibniz identity, while the Hochshild 1-coboundaries are inner derivations, i.e., maps* $A \to M$ *of the form* $a \mapsto am - ma$ *for some* $m \in M$*. So* $\mathrm{HH}^1(A,M)$ *is the quotient of the two, the so called space of outer derivations.*

The groups that are relevant for the deformation theory are mostly $\mathrm{HH}^2(A)$ and $\mathrm{HH}^3(A)$.

The definition of the Hochschild cohomology may look strange. However, this is a special case of a more general construction, the Ext groups. Let us recall some generalities.

Let B be an associative algebra. A B-module P is called projective, if it is a direct summand of a free B-module. An alternative characterization: P is projective if the functor $\mathrm{Hom}_B(P,\bullet)$ is exact. For a B-module M we have a projective (for example, free) resolution $\to P_i \to P_{i-1} \to \ldots \to P_0$. Then given another B-module N, we can form the Ext groups $\mathrm{Ext}^i(M,N)$, by definition $\mathrm{Ext}^i(M,N)$ is the ith cohomology group of the complex $\mathrm{Hom}(P_0,N) \to \mathrm{Hom}(P_1,N) \to \ldots$. This definition is independent of the choice of a resolution.

Lemma 6.1. $\mathrm{HH}^i(A,M) = \mathrm{Ext}^i_{A-Bimod}(A,M)$.

Proof. Consider a free resolution of the A-bimodule A (called the standard resolution). It is given by $P_i = A^{\otimes i+2}$, where we view $A^{\otimes i+2}$ as a bimodule using external actions: $b(a_1 \otimes \ldots \otimes a_{i+2})c = ba_1 \otimes a_2 \otimes \ldots a_{i+1} \otimes a_{i+2}c$, for a basis of the A-bimodule $A^{\otimes i+2}$ we can take the basis of $1 \otimes A^{\otimes i} \otimes 1$ over $\mathbb{C}$. The differential is defined by

$$d(a_1 \otimes \ldots \otimes a_{i+2}) := a_1 \otimes a_2 \otimes \ldots \otimes a_{i+1}a_{i+2} - a_1 \otimes a_2 \otimes \ldots \otimes a_i a_{i+1} \otimes a_{i+2} + \ldots.$$

To see that we indeed get a resolution one considers the homotopy $h : A^{\otimes i+1} \to A^{\otimes i+2}$ given by $h(x) = 1 \otimes x$.

We can identify $C^n(A,M)$ with $\mathrm{Hom}_{A-Bimod}(A^{\otimes n+2}, M)$ by sending $\varphi \in \mathrm{Hom}_{A-Bimod}(A^{\otimes n+2}, M)$ to the map $\Phi : A^{\otimes n} \to M$ given by $\Phi(x) = \varphi(1 \otimes x \otimes 1)$. It is easy to see that the Hochschild differential becomes identified with one induced from the standard resolution. ☐

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