LECTURES ON SYMPLECTIC REFLECTION ALGEBRAS

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9. Commutativity and centers

9.1. **Commutativity theorem: statement and scheme of proof.** It is a natural question to ask when the algebra $eH_{t,c}e$ is commutative. This happens to have a very elegant answer that was already stated in Lecture 5 in the case when $\dim V=2$.

Theorem 9.1 ([EG])**.** *The algebra $eH_{t,c}e$ is commutative if and only if $t=0$.*

We will give a proof in the case when V is symplectically irreducible. The proof will be in three steps.

Step 1. We have the bracket $\{\cdot,\cdot\}_{t,c}$ of degree -2 on $S(V)^\Gamma$ induced by the filtered deformation $eH_{t,c}e$. We will see that this bracket depends on t and c linearly. The reason why we are interested in considering the bracket $\{\cdot,\cdot\}_{t,c}$ is that this bracket is zero when $eH_{t,c}e$ is commutative.

Step 2. So we have a linear map from P^* to the space of brackets of degree -2 on $S(V)^\Gamma$. We will see that such a bracket on $S(V)^\Gamma$ is unique up to proportionality provided V is symplectically irreducible. Thanks to this we will have a hyperplane $P_0\subset P^*$ such that for $p=(t,c)\in P_0$ we have $\{\cdot,\cdot\}_p=0$.

A priori it may be that the algebra eH_pe is not commutative although $\{\cdot,\cdot\}_p=0$. In this case, $eH_{t,c}e$ still gives rise to a non-zero bracket on $S(V)^\Gamma$ but the degree of that bracket will be $-d$ with $d>2$. We will see however that all brackets on $S(V)^\Gamma$ of that degree are 0. So for $(t,c)\in P_0$ the algebra $eH_{t,c}e$ is indeed commutative.

Step 3. It remains to show that P_0 is given by $t=0$. For this it is enough to produce a representation of $H_{t,c}$ isomorphic to $\mathbb{C}\Gamma$ as a Γ-module. Indeed, if N is such a representation, then we have $0=\operatorname{tr}_N[u,v]=\operatorname{tr}_N(\omega(u,v)t+\sum_{s\in S}c_s\omega_s(u,v)s)=t\dim N\omega(u,v)$. In fact, we will show that for N one can take $H_{t,c}e/\mathfrak{m}$, where $\mathfrak{m}$ is a generic maximal ideal of $eH_{t,c}e$.

9.2. **Step 1.** We write p for $(t,c)\in P^*$, $\mathcal{A}$ for eHe, and $\mathcal{A}_p$ for $eH_{t,c}e$.

Lemma 9.2. (1) *We have $[a,b]\in P\mathcal{A}$ for $a,b\in\mathcal{A}$.*
(2) *$[\mathcal{A}^i,\mathcal{A}^j]\subset P\mathcal{A}^{i+j-2}$,*
(3) *and $[\mathcal{A}_p^{\leqslant i},\mathcal{A}_p^{\leqslant j}]\subset\mathcal{A}_p^{i+j-2}$.*

Proof. (1) follows from $\mathcal{A}/P\mathcal{A}\cong S(V)^\Gamma$ because $S(V)^\Gamma$ is commutative. (2) follows from (1) and the condition that the degree of P is 2. (3) follows from (2). □

So we do have a bracket $\{\cdot,\cdot\}_p$ of degree -2 on $S(V)^\Gamma$. Moreover, it is obtained as follows: take homogeneous $a_0,b_0\in S(V)^\Gamma$. Then let a,b be homogeneous elements in eHe lifting a_0,b_0. Then $[a,b]\in P\mathcal{A}$. In particular, we can consider the projection $\{a_0,b_0\}$ of $[a,b]$ to $P\otimes S(V)^\Gamma=P\mathcal{A}/P^2\mathcal{A}$ (recall that $\mathcal{A}=S(P)\otimes S(V)^\Gamma$ as an $S(P)$-module). Then $\{a,b\}_p$ is a specialization of $\{a,b\}$ at $p:P\to\mathbb{C}$. This implies the claim on linearity.

9.3. **Step 2.** We claim that every bracket on $S(V)^\Gamma$ uniquely lifts to a Γ-invariant bracket on $S(V)$. To prove this we need to explore a geometric nature of brackets.

Let X be an affine algebraic variety. Suppose that $\mathbb{C}[X]$ is equipped with a bracket $\{\cdot,\cdot\}$. Pick a smooth point $x \in X$. Then we can define a bivector $\mathcal{P}_x \in \bigwedge^2 T_x X$ by setting $\langle \mathcal{P}_x, df \wedge dg\rangle = \{f,g\}(x)$. It is easy to see that this is well-defined. A bit more subtle observation (that is consequence of the fact that the tangent and cotangent sheaves on a smooth variety are locally free) is that the bivectors $\mathcal{P}_x$ glue together to form a section $\mathcal{P}$ of $\bigwedge^2 TX^{reg}$, where X^{reg} denotes the smooth locus of X. Now assume that the variety X is normal so that, in particular, $\mathbb{C}[X] = \mathbb{C}[X^{reg}]$. Then a bivector $\mathcal{P} \in \Gamma(X^{reg}, \bigwedge^2 TX^{reg})$ gives rise to a bracket on $\mathbb{C}[X^{reg}] = \mathbb{C}[X]$ – by $\{f,g\} = \langle \mathcal{P}, df \wedge dg\rangle$.

Another fact about brackets that we need is that a bracket can be pulled back by an étale morphism. Recall that a morphism $\varphi : Y \to X$ of smooth varieties is called étale at $y \in Y$ if $d_y\varphi$ is an isomorphism. We say that φ is étale if it is étale at any point. So we can identify $T_yY \cong T_{\varphi(x)}X$ and therefore also $\bigwedge^2 T_yY$ with $\bigwedge^2 T_{\varphi(y)}X$. A more subtle claim again, is that one can pull-back $\mathcal{P} \in \Gamma(X, \bigwedge^2 TX)$ to get a well-defined element $\varphi^*(\mathcal{P}) \in \Gamma(Y, \bigwedge^2 TY)$ (this follows from $TY = \varphi^*(TX)$).

Finally, we need to characterize $(V/\Gamma)^{reg}$ and find the locus, where the quotient morphism $\pi : V \to V/\Gamma$ is étale. This is explained in the following lemma, where we assume that Γ is just some finite subgroup of $\mathrm{GL}(V)$. Recall the notation $V^0 = \{v \in V | \Gamma_v = 0\}$.

Lemma 9.3. *We have $V^0/\Gamma \subset (V/\Gamma)^{reg}$. The morphism π is étale at all points of V^0.*

We are not going to prove the lemma. It is clear if we work in the complex analytic, not algebraic category. It also fixes a gap in the proof of a technical lemma of Step 2 in the proof of the double centralizer property. Finally, let us remark that if Γ contains no *complex* reflections (this is always the case for $\Gamma \subset \mathrm{Sp}(V)$), then the inclusions in the lemma are actually equalities.

Now we are ready to prove Step 2. Let $\{\cdot,\cdot\}$ be a bracket on $S(V)^\Gamma \cong \mathbb{C}[V]^\Gamma$ and let $\mathcal{P}$ be a corresponding bivector on $V^0/\Gamma \subset (V/\Gamma)^{reg}$. Then we get a bivector $\pi^*(\mathcal{P})$ on V^0 and hence a bracket $\{\cdot,\cdot\}'$ on $\mathbb{C}[V^0] = \mathbb{C}[V]$. The bivector and hence the bracket are Γ-equivariant by construction. Also by construction, the restriction of $\{\cdot,\cdot\}'$ to $\mathbb{C}[V]^\Gamma$ coincides with $\{\cdot,\cdot\}$ and $\{\cdot,\cdot\}'$ is a unique Γ-equivariant (the latter is not necessary) bracket with these properties. From here it follows that the degree of $\{\cdot,\cdot\}'$ is the same as that of $\{\cdot,\cdot\}$ (if $\{\cdot,\cdot\}'$ has components of other degrees, then they restrict to 0 on $\mathbb{C}[V]^\Gamma$).

So now the question is: describe Γ-equivariant brackets of degree $\leqslant -2$ on $\mathbb{C}[V]$. If the degree of $\{\cdot,\cdot\}'$ is less than -2, then this bracket vanishes on V^*, the degree 1 component of $\mathbb{C}[V]$. So $\{\cdot,\cdot\}'$ is identically 0. Similarly, the bracket of degree -2 just comes from a skew-symmetric form on $V^* \cong V$. This form is Γ-invariant. If V is symplectically irreducible, then there is a unique such form up to proportionality.

Let us remark that for some p we do have $\{\cdot,\cdot\}_p \neq 0$. Indeed, consider the case when $c = 0, t = 1$. Then $H_{t,c} = T(V)\#\Gamma/(u\otimes v - v\otimes u - \omega(u,v)) = W(V)\#\Gamma$ so that $eH_{t,c}e \cong W(V)^\Gamma$. As we have seen, the bracket on $S(V)$ induced by $W(V)$ coincides with the standard bracket. It follows that the bracket on $S(V)^\Gamma$ induced by $W(V)^\Gamma$ also coincides with the standard bracket on hence is nonzero.

So we have a hyperplane $P_0 \subset P^*$ such that $\{\cdot,\cdot\}_p = 0$ for all $p \in P_0$ and hence $\mathcal{A}_p$ is commutative.

Exercise 9.1. *Show that $\{\cdot,\cdot\}_{t,c} = t\{\cdot,\cdot\}$, where $\{\cdot,\cdot\}$ is the standard bracket on $S(V)^\Gamma$.*

Exercise 9.2. *Prove the commutativity theorem in the case when V is not necessarily symplectically irreducible.*

9.4. **Step 3.** According to the next exercise $\mathcal{A}_p$ is always finitely generated.

Exercise 9.3. *Let $\mathcal{A}$ be a $\mathbb{Z}_{\geqslant 0}$-filtered algebra. If $\operatorname{gr}\mathcal{A}$ is finitely generated, then so is $\mathcal{A}$.*

So for $p \in P_0$ one can consider $C_p := \operatorname{Spec}(\mathcal{A}_p)$ (a problem below implies that $\mathcal{A}_p$ has no zero divisors so that C_p is an irreducible variety). Recall that we have commuting actions of H_p and $\mathcal{A}_p$ on H_pe. As we have seen, it is enough to prove the following claim. Let $p \in P_0$, then for a general point $x \in C_p$ the H_p-module $H_pe/\mathfrak{m}_x$ is isomorphic to $\mathbb{C}\Gamma$ as a Γ-module. This amounts to checking that for any Γ-irreducible L the dimension of the fiber of $M_p^L := \operatorname{Hom}_\Gamma(L, H_pe)$ at a general point $x \in C_p$ equals $\dim L$ (here the action of $\mathcal{A}_p$ on $\operatorname{Hom}_\Gamma(L, H_pe)$ is induced from the action of $\mathcal{A}_p$ on H_pe from the right). We remark that this holds for $p = 0$: for $v \in V_0$ we have $\mathbb{C}[V]_{\pi(v)} = \mathbb{C}[\Gamma v] \cong \mathbb{C}\Gamma$.

Pick a nonzero parameter $p \in P_0$. To prove our claim we will need to include C_p and V/Γ into a single variety. For this let R be the one-dimensional subspace of P^* spanned by p. Consider the algebras $H_R := \mathbb{C}[R] \otimes_{S(P)} H, \mathcal{A}_R := eH_Re$. Then $\mathcal{A}_R$ is a deformation of $S(V)^\Gamma$ over $\mathbb{C}[R]$ and hence $\mathcal{A}_R$ is finitely generated. Set $C_R := \operatorname{Spec}(\mathcal{A}_R)$. This is an algebraic variety equipped with a $\mathbb{C}^\times$-action coming from the grading on $\mathcal{A}_R$ and also a $\mathbb{C}^\times$-equivariant morphism $C_R \to R$, whose zero fiber is V/Γ, while a nonzero fiber is naturally identified with C_p (the fiber over p is literally C_p, and all other fibers are translated to C_p using the $\mathbb{C}^\times$-action). The $\mathcal{A}_R$-module H_Re is flat over $\mathbb{C}[R]$. Consider the module $M_R^L := \operatorname{Hom}_\Gamma(L, H_Re)$. Let $\mathfrak{p}$ denote the ideal of $\mathcal{A}_R$ generated by R^*. Since $\mathcal{A}_R/\mathfrak{p} \cong \mathbb{C}[V]^\Gamma$, the ideal $\mathfrak{p}$ is prime. So we can consider the localization $\mathcal{A}_{R,\mathfrak{p}}$, a local ring of dimension 1 with a local parameter r, a basis element in R^*, and residue field $\mathbb{C}(V/\Gamma)$. We know that the localization $M_{R,\mathfrak{p}}^L$ is flat over $\mathbb{C}[r]$ and the fiber at $r = 0$ has dimension $\dim L$ over $\mathbb{C}(V/\Gamma)$. What we need to prove is that the dimension of the localization $\mathbb{C}(C_R) \otimes_{\mathcal{A}_R} M_R^L$ equals $\dim L$ (this implies existence of a required point x on a general fiber of $C_R \to R$ and, therefore, thanks to $\mathbb{C}^\times$-equivariance, on an arbitrary nonzero fiber). Thanks to flatness, $\dim_{\mathbb{C}(C_R)} \mathbb{C}(C_R) \otimes_{\mathcal{A}_R} M_R^L = \dim_{\mathbb{C}(V/\Gamma)} M_{R,\mathfrak{p}}^L/(r)$. The latter coincides with $\dim L$.

This completes the proof of the commutativity theorem.

9.5. **Satake isomorphism.**

Theorem 9.4 ([EG]). *Let Z_c be the center of $H_{0,c}$. The map $z \mapsto ez$ is an isomorphism of Z_c and $eH_{0,c}e$.*

Proof. We are going to find an inverse homomorphism. We write p for $(0, c)$. Recall that the action of H_p on H_pe gives rise to an isomorphism $H_p \xrightarrow{\sim} \operatorname{End}_{\mathcal{A}_p}(H_pe)$. Since the algebra $\mathcal{A}_p$ is commutative the map $m \mapsto mb$ is an endomorphism of the $\mathcal{A}_p$-module H_pe for any $b \in \mathcal{A}_p$. Such an endomorphism commutes with any other. Let $\hat{b}$ denote a unique element of $H_p = \operatorname{End}_{\mathcal{A}_p}(H_pe)$ such that $\hat{b}m = mb$. Clearly, $b \mapsto \hat{b}$ is an algebra homomorphism $\mathcal{A}_p \to Z_p$. The claim that this homomorphism is inverse to $z \mapsto ez$ means $\widehat{ez} = z, e\hat{b} = b$. We have $\widehat{ez}m = mez = mz = zm$ and so $\widehat{ez} = z$. On the other hand, $me\hat{b} = m\hat{b} = \hat{b}m = mb$. Plugging $m = e$, we get $e\hat{b} = eb = b$. $\square$

Problem 9.4. *Let $p \in P_0$. Equip Z_p with a filtration restricted from H_p. Show that $\operatorname{gr} Z_p = S(V)^\Gamma$. Deduce that H_p is a finitely generated module over Z_p.*

Problem 9.5. *Now let $p \notin P_0$. Show that the center of H_p coincides with $\mathbb{C}$ as follows:*

(1) *Let* z *lie in the center of* H_p. *Show that* $\operatorname{gr} z \in \operatorname{gr} H_p = S(V)\#\Gamma$ *actually lies in* $S(V)^\Gamma$.
(2) *Show that* $\operatorname{gr} z$ *lies in the Poisson center of* $S(V)^\Gamma$, *meaning that* $\{\operatorname{gr} z, S(V)^\Gamma\} = 0$.
(3) *Show that the Poisson center of* $S(V)^\Gamma$ *coincides with* $\mathbb{C}$.

Problem 9.6. *In this problem we are going to equip* Z_c *with a structure of a Poisson algebra. Fix* c *and consider* $H_{t,c}$ *as an algebra over* $\mathbb{C}[t]$ *by making* t *an independent variable.*

(1) *Let* $a, b \in Z_c$. *Lift* $a, b \in H_c = H_{t,c}/(t)$ *to elements* $\tilde{a}, \tilde{b} \in H_{t,c}$. *Show that* $[\tilde{a}, \tilde{b}] \in tH_{t,c}$ *and that the element* $\frac{1}{t}[\tilde{a}, \tilde{b}]$ *modulo* t *depends only on* a, b *and lies in* Z_c. *Let* $\{a, b\}$ *be that element. Show that* $\{\cdot, \cdot\}$ *is the Poisson bracket.*
(2) *Show that* $\{Z_c^{\leqslant i}, Z_c^{\leqslant j}\} \subset Z_c^{i+j-2}$. *Show that the induced bracket on* $\operatorname{gr} Z_c = S(V)^\Gamma$ *is a nonzero multiple of the standard bracket. Can you identify the scalar factor?*

Problem 9.7. *Show that the scheme* C_p *is irreducible and normal (and, well, Cohen-Macaulay and Gorenstein, if you know what these words mean).*

9.6. **Further algebraic properties.** Perhaps, the first question about the structure of an irreducible normal (and also Cohen-Macaulay and Gorenstein) algebraic variety you can ask is whether it is smooth.

First of all, one can describe the smooth points $x \in C_p$ in terms of the representation theory of the algebra $H_p/H_p\mathfrak{m}_x$.

Theorem 9.5 ([EG])**.** *The following are equivalent.*

(1) $x \in C_p^{reg}$.
(2) $H_p/H_p\mathfrak{m}_x \cong \operatorname{End}(\mathbb{C}\Gamma)$ *(a* Γ*-equivariant algebra isomorphism).*
(3) *Any simple* $H_p/H_p\mathfrak{m}_x$*-module is isomorphic to* $\mathbb{C}\Gamma$ *as a* Γ*-module.*

Problem 9.8. *Show that if* C_p *is smooth, then* H_pe *is a locally free* H_p*-module.*

The proof is based on properties of PI (polynomial identity) rings and we are not going to provide it.

Let us explain what is known about smoothness of the varieties C_p. First, of all existence of p such that C_p is smooth is a very restrictive assumption on Γ.

If $\Gamma = \Gamma_n$ is a wreath-product $\mathfrak{S}_n \ltimes \Gamma_1^n$, where $\Gamma_1 \subset \mathrm{SL}_2(\mathbb{C})$, then C_p is smooth if and only if p lies outside the union of explicitly described hyperplanes. This follows from the interpretation of C_p as affine quiver varieties to be covered later in this course.

In the class of complex reflection groups, the answer is also known. Besides the groups $G(\ell, 1, n)$ that belong also to the previous list, there is only one complex reflection group such that there is a smooth C_p. It is the group G_4 that appeared in one of the problems of Lecture 6. This was proved by Bellamy in [B].

The situation with the groups that do not belong to one of these families is unknown. Recently, Bellamy and Schedler, [BS], found an example of such a group that does admit smooth C_p.

This question has to do with sphericity of parameters discussed last time. Namely, C_p is smooth iff p is spherical (meaning that $H_peH_p = H_p$). Indeed, the smoothness of $C_p = \operatorname{Spec}(\mathcal{A}_p)$ is equivalent to $\mathcal{A}_p$ having finite global dimension. The following problem implies that H_p has finite global dimension (for an arbitrary $p \in P^*$).

Problem 9.9. *Let* $\mathcal{A}$ *be a filtered algebra. Show that if* $\operatorname{gr}\mathcal{A}$ *has finite global dimension, then* $\mathcal{A}$ *does.*

As we have discussed earlier, $S(V)\#\Gamma$ has finite global dimension (equal $\dim V$) and so the global dimension of H_p is finite. Now the global dimension is an invariant of the category of modules, and $H_p e H_p = H_p$ means that the categories of modules for H_p and eH_pe are equivalent. So if p is spherical, then C_p is smooth.

Problem 9.10. *Conversely, prove that if C_p is smooth, then p is spherical (we deal here with $p \in P_0$).*

In fact, $p \in P^*$ is spherical iff the global dimension of eH_pe is finite even if the latter is not commutative. This is a result of Bezrukavnikov, [E2].

References

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