PROBLEMS ON SYMPLECTIC REFLECTION ALGEBRAS

10. Moment maps in algebraic setting

Exercise 10.1. *Let A be a commutative algebra and B be a localization of A. Show that any bracket on A extends to a unique bracket on B.*

Exercise 10.2. *Show that the Poisson bracket on $\mathbb{C}[T^*X_0]$ can be characterized as follows: we have $\{f,g\}=0, \{\xi,f\}=L_\xi f, \{\xi,\eta\}=[\xi,\eta]$ for $f,g\in\mathbb{C}[X_0], \xi,\eta\in \mathrm{Vect}(X_0)$. Deduce that, with respect to the standard grading on $\mathbb{C}[T^*X_0]=S_{\mathbb{C}[X_0]}(\mathrm{Vect}(X_0))$, the bracket has degree -1.*

Exercise 10.3. *Show that a G-equivariant map $\xi\mapsto H_\xi$ with $v(H_\xi)=\xi_X$ is automatically a Lie algebra homomorphism.*

Exercise 10.4. *Let μ,μ' be two moment maps, and X be connected. Then $\mu-\mu'$ is a constant map equal to some G-invariant element of $\mathfrak{g}^{*G}$.*

Exercise 10.5. *Prove that $\ker d_x\mu = \mathfrak{g}x^\perp$ and $\operatorname{im} d_x\mu=\mathfrak{g}_x^\perp$, where in the first equality the superscript $\perp$ stands for the skew-orthogonal complement with respect to ω_x, and in the second case for the annihilator in the dual space; we write $\mathfrak{g}_x$ for the Lie algebra of stabilizer G_x. Deduce that $d_x\mu$ is surjective if and only if G_x is discrete.*

Exercise 10.6. *Let $\mu: T^*X_0\to\mathfrak{g}^*$ be the moment map. Show that $\mu^{-1}(0)$ is the union of conormal bundles to the G-orbits in X_0.*

Problem 10.1. *Let G act on a vector space V with finitely many orbits. Show that G acts on V^* with finitely many orbits and exhibit a bijection between the two sets of orbits.*

Problem 10.2. *Let V be a symplectic vector space with form ω and let G act on V via a homomorphism $G\to \mathrm{Sp}(V)$. Show that this action is Hamiltonian with $H_\xi(v)=\frac{1}{2}\omega(\xi v,v)$.*

Problem 10.3. *This problem discusses symplectic forms on coadjoint orbits. Let G be an algebraic group. Pick $\alpha\in\mathfrak{g}^*$.*

1. *Equip $T_\alpha G\alpha$ with a form ω_α by setting $\omega_\alpha(\xi_\alpha,\eta_\alpha)=\langle\alpha,[\xi,\eta]\rangle$. Prove that this is well-defined.*
2. *Show that ω_α extends to a unique G-invariant form on $G\alpha$ (the Kirillov-Kostant form) and that this form is symplectic. Further, show that the G-action on $G\alpha$ is Hamiltonian with moment map being the inclusion.*
3. *Let X be a homogeneous space for G equipped with a symplectic form ω such that the G-action is Hamiltonian with moment map μ. Show that the image of μ is a single orbit, say $G\alpha$, that μ is a locally trivial covering, and that ω is obtained as the pull-back of the Kirillov-Kostant form.*