PROBLEMS ON SYMPLECTIC REFLECTION ALGEBRAS

15. Quotient singularities as quiver varieties

The purpose of this problem set is to recover results of Lecture 15 in the special case of $\Gamma_1 = \{1\}$.

Problem 15.1. *Describe the* $\mathrm{GL}(n)$*-orbits on* $\mathrm{Mat}_n(\mathbb{C}) \oplus \mathbb{C}^n$. *Deduce that there are* $n+1$ *components in* $\mu^{-1}(0)$ *and that they all have codimension* n^2.

As in the correction to the lecture, we show that $\mu^{-1}(0)$ is reduced. So $\mu^{-1}(0)//G$ is a variety.

Problem 15.2. *This problem establishes a bijective morphism* $\psi : \mathbb{C}^{2n}/\mathfrak{S}_n \to \mu^{-1}(0)//G$.

(1) *Let* $A, B \in \mathrm{Mat}_n(\mathbb{C})$ *be such that* $\mathrm{rk}[A,B] \leqslant 1$. *Show that, in some basis,* A, B *are upper triangular.*
(2) *Deduce that any irreducible representation of* $\Pi^0(Q^{CM})$ *is actually a one-dimensional representation of* $\Pi^0(Q^{MK})$.
(3) *Use this to produce a required morphism.*

It remains to show that ψ^* is isomorphism of algebras.

Problem 15.3. *Show that* $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]^{\mathfrak{S}_n}$ *is generated by the polynomials of the form* $\sum_{i=1}^n (x_i + t y_i)^n$. *Deduce that* ψ^* *is surjective and, using this, that* ψ^* *is an isomorphism.*